Constructing Static Black Hole-Soliton Spacetimes

by

© **Andrew M. Rose**

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Abstract

The purpose of this work is to construct new static solutions of five-dimensional vacuum general relativity describing asymptotically flat black hole-soliton spacetimes. An asymptotically flat soliton is a globally stationary, everywhere regular positive-energy solution. They are characterized by non-trivial spacetime topology. It can be proved that in the vacuum everywhere-regular solitons cannot exist and we explicitly show this in specific examples by studying conical singularities in the spacetime. However, the existence of spacetimes containing both black holes *and* solitons has not been ruled out. To investigate this problem in a simplified setting, this thesis will focus on static solutions with two rotational symmetries. These solutions are known as Weyl solutions. We construct explicit solutions of this type and study their properties. We then generalize our solutions to the case with non-vanishing Maxwell fields and obtain static, electrically charged, black hole-soliton spacetimes. Throughout the work we take a general approach that should be valuable in a wider setting.

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Statement of contribution

This work was base on a research problem suggested by the Author's supervisor Dr. Hari Kunduri. All the work and writing conducted in this thesis was completed by the Author apart from some calculations in Section 4.3.2 which were completed in collaboration with the supervisor.

Table of contents

List of figures

Chapter 1

Introduction

Einstein's theory of General Relativity [4] is perhaps the single best known mathematical theory describing the natural world in existence. This theory provides a framework to define the structure of spacetime. In this way, we may assign a mathematical formulation of gravitation and how it acts in the Universe from a massive scale to the infinitesimal.

In general relativity and black hole physics there is a well-known theorem known as the "no hair" theorem [3]. This theorem loosely states that a 4 dimensional, stationary, asymptotically flat black hole can be entirely characterized by its mass, M, its electric charge, Q, and its angular momentum, J. This theorem is a statement of uniqueness, that black hole solutions to the Einstein-Maxwell equations with the same properties M, Q, and J are identical [13, 20]. More specifically, these black holes must belong to the Kerr-Newmann family of solutions. For a more modern analysis of these uniqueness properties see [3].

In this work, we are concerned with the solutions found in 5 dimensional spacetimes. This is of great interest because the leading candidate for a theory of quantum gravity, string theory, asserts the existence of more than 3 spatial dimensions [7]. In addition, developments in modern theoretical physics, such as the gauge theory-gravity correspondence, points us in the direction that higher dimensional spacetimes are key to properly understanding our universe. Finally, this work is of interest in a purely mathematical sense. Many advances in this field of research yield valuable results to the mathematical community at large. In particular, results in this area motivated advances in geometric analysis, such as the positive mass theorem [24].

The idea of *soliton* solutions are important to this work [1]. *Solitons* describe isolated, stationary, self-gravitating systems that have finite energy but are not black holes (i.e. they do not contain event horizons). They can arise as solutions to the Einstein-Maxwell field equations [4]. In Einstein-Maxwell theory 4 dimensional soliton spacetimes do not exist because the spacetime must be simply-connected [27]. However, examples are known to exist in Einstein-Yang-Mills theory, see [25]. In particular, using Stokes' Theorem [17] and a variety of mathematical identities it can be proved that the only way a stationary spacetime has positive energy is if it contains a black hole.

However, in the case of greater than 4 dimensions, in particular D = 5, this is not necessarily the case. In the case of 5 dimensions it may be that there exists so called "bubbles" or 2-cycles. 2-cycles are 2 dimensional closed surfaces that are not the boundary of a 3 dimensional volume, they are simply "bubbles". These 2-cycles can be prevented from collapsing due to gravity by introducing magnetic flux from a non-vanishing Maxwell field. This is only possible in greater than 4 dimensional spacetimes. In 4 dimensions it can be proved that these "bubbles" will simply collapse in an asymptotically flat spacetime. These isolated systems can carry positive energy without the need for a black hole [10, 17]. These 2-cycles can provide information, in particular an observer can measure mass, electric charge, and/or angular momentum that we physically interpret as associated to the bubbles. Note that the only static, asymptotically flat vacuum solutions containing a black hole in n > 3 dimensions must be Schwarzschild in nature [9].

Asymptotically flat soliton spacetimes are difficult to construct, and as stated above they cannot exist in the vacuum or in Einstein-Maxwell theory. For concreteness we give a simple example of a spacetime containing a 2-cycle, although it is not asymptotically flat. Take the vacuum solution consisting of a flat time direction times the Schwarzschild metric with imaginary time, given by

$$ds^2 = -dt^2 + (1 - \frac{2M}{r})d\psi^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where ψ is now an angular coordinate. This is a Ricci flat metric, i.e. $R_{ab} = 0$. By analyzing the metric carefully one can show that regularity (removal of conical singularities) requires the ψ to have period $\frac{1}{8\pi M}$. There is a 2-cycle at $r = 2M$ because the (r,ψ) part of the metric degenerates to the origin of R^2 while there is a 2-sphere of radius $2M$, represented in the (θ, ϕ) part of the metric. The topology of the manifold is $R \times R^2 \times S^2$. Asymptotically, however it is not flat but rather asymptotically Kaluza-Klein, i.e. as $r \to \infty$, the metric tends to

$$ds^2 = -dt^2 + dr^2 + d\psi^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which is a metric on $R \times S^1 \times R^3$. This focus of this thesis is concerned with finding solutions of the Einstein equations similar to the above metric, but which are actually asymptotically flat instead of asymptotically Kaluza-Klein.

With this in mind, we know that the properties (M, Q, J) are insufficient to fully characterize a black hole in 5 dimensions. As it stands now it is not possible for an observer at a distance to determine if they are observing a black hole with a particular mass, electric charge, or angular momentum or if they are observing the effect of 2-cycles. In fact, this may occur without the presence of a black hole at all. Since, to an outside observer, these systems can all look the same we have lost a sense of uniqueness. In fact, it is expected that spacetimes containing both solitons and black holes exist, even in vacuum space (however in light of the vacuum uniqueness theorem [9], we know such solutions would have to be non-static). Such spacetimes are characterized by having both 2-cycles and event horizons in the regions surrounding black holes. This adds to the difficulty of defining uniqueness.

In this work we will attempt to understand the properties of these systems. This will be in the form of finding solutions to various examples of 5 dimensional solutions to the Einstein-Maxwell equations. It is known that few such examples exist [16] so it would be of great benefit to study these. While the results of static vacuum systems are well known, we can search for results to *charged* systems, or use the smooth, static solutions as "seeds" to generate solutions to the rotating system [7].

Chapter 2 will provide the background of our problem to be studied. This will involve the introduction of Weyl solutions, the basics of rod diagrams, and the structure of the metric in this spacetime. Basic transformations are introduced for future use in modifying the problem statements for analysis. We also introduce conical singularities and the mathematical language used to describe them.

In chapter 3 will involve the methodology for solving the associated coupled PDEs for the examples we have designed to be studied. This involves deriving techniques for solving these systems. The solution will create the metric as desired. In addition, we will study the associated conical singularities and the issues found in designing consistent spacetimes.

For chapter 4 we will conduct a series of transformations on a generic metric in order to write it in a form that is amenable to study. This will include properties such as the electric charge, mass, surface gravity, and the associated spacetime fields. These boosts take into account string theoretic mathematics to account for the 5 dimensional analysis. Chapter 5 will provide a summary and analysis of the results determined throughout the work.

These determinations are useful as this gives a result for 5 dimensional spacetimes while accounting for string theory and quantum gravity in the calculations. This work is highly motivated by the work conducted by Dr. Hari Kunduri and Dr. James Lucietti as well as the work by Dr. Roberto Emparan, Dr. Harvey Reall, and Dr. Henriette Elvang. While this research was being conducted a further paper by Kunduri and Lucietti was published eliminating static Einstein-Maxwell black hole soliton solutions [18]. All computations were completed using the Mathematica software package.

Chapter 2

Background Review and Problem Statement

2.1 Basics of General Relativity

The most important mathematical object in the study of General Relativity is the metric tensor. This can be considered in two forms: as a tensor or as a line element. As a tensor, g, the metric is the tensor

$$g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} \tag{2.1}$$

expressed relative to the coordinate fields x^{ν}. Here $g_{\mu\nu}$ is the set of coefficients corresponding to these tensor products. This product is symmetric, thus $g_{\mu\nu} = g_{\nu\mu}$. The line element form takes on a similar structure, typically written as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{2.2}$$

In most of our analysis we will be using the line element form of the metric, however it is useful to consider the tensor form as it will be useful for manipulations. This form of the metric emphasizes its role as measuring the squared distance between two points separated by coordinate distance dx^μ. As an example, we may consider Minkowski Spacetime. This is also referred to as "Flat" spacetime due to the lack of curvature from massive objects. In this case, the line element form of the metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.3}$$

where t is the time component of the tensor and the x, y, z terms are the typical 3 Dimensional physical directions.

There are a number of key properties that we are concerned with in our analysis. In particular the standard objects in the theory such as the christoffel symbols, curvature tensor, Ricci tensor, and field equations are important. The general form for the Christoffel Symbols is given by

$$\Gamma_{cab} = \frac{1}{2}\left(\frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^c}\right) = \frac{1}{2}\left(\partial_b g_{ca} + \partial_a g_{cb} - \partial_c g_{ab}\right) . \tag{2.4}$$

while the Ricci tensor is given by

$$R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta} = \partial_\rho \Gamma^\rho{}_{\beta\alpha} - \partial_\beta \Gamma^\rho{}_{\rho\alpha} + \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\beta\alpha} - \Gamma^\rho{}_{\beta\lambda}\Gamma^\lambda{}_{\rho\alpha} \tag{2.5}$$

where

$$g^{cd}\Gamma_{cab} = \Gamma^d{}_{ab}. \tag{2.6}$$

In general, the Einstein Field Equations can be written as

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab} \tag{2.7}$$

where R_{ab} is the Ricci tensor, R is the Ricci Scalar, g_{ab} is the element of the metric, and T_{ab} is the stress-energy tensor. In the case of a vacuum, $T_{ab} = 0$, thus we are left with the simplified form

$$R_{ab} = \frac{1}{2} R g_{ab} \tag{2.8}$$

Thus, if we take the trace of (2.8) we have that

$$\mathbf{Tr}(R_{ab}) = \frac{1}{2} R \mathbf{Tr}(g_{ab}) \tag{2.9}$$

Since, in general, $\mathbf{Tr}(g_{ab}) = D$, the dimension, we have the result $R = R_{ab} = 0$ [21]. A specific way to write the Einstein-Maxwell field equations in $D = 5$, which will be useful for our purposes, is

$$\begin{aligned} R_{\mu\nu} &= 2\left(F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \\ \nabla_{\mu}\left(F^{\mu\nu}\right) &= 0 \end{aligned} \tag{2.10}$$

2.2 4D Black Holes

The case of a typical black hole solution will be considered in 4 dimensions as this is representative as the baseline case of a spacetime solution.

2.2.1 4 Dimensional Schwarzschild

The general form of the metric for the static, spherically symmetric 4 dimensional spacetime in spherical coordinates is given by

$$ds^2 = -U(r)dt^2 + V(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.11}$$

We can see that for the flat solution computation shows that the curvature is identically zero. The most general vacuum solution of the form (2.11) with $U(r), V(r) \to 1$ as $r \to \infty$ yields the familiar result

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (2.12)$$

This metric describes a black hole. As $r \to \infty$ the metric approaches the previously described flat space. As $r \to 0$ we obtain a singularity. Finally, as $r \to 2GM$, it appears to be singular, but further analysis yields that this is a hypersurface, in particular this is the event horizon corresponding to a Schwarzschild black hole. The parameter M can be shown to be related to the energy of the spacetime.

2.2.2 Kerr Black Hole

Kerr black holes are the only known asymptotically flat, stationary, non-static black hole solutions of Einstein's equation [26]. In fact, it has been shown that the Kerr solutions are the only *possible* stationary, vacuum, black hole solutions. A result of this is the fact that if the spacetime surrounding a gravitational collapse results in a stationary vacuum the associated black hole will always be a Kerr type [26]. This uniqueness proof has been detailed by Chruściel et al [3] and is left to the reader. Other works by Carter [2] and Robinson [23] are useful in the full details and understanding of this result. Thus, the only known stationary, non-static black hole solutions must take on the form

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned} \qquad (2.13)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$ [26]. Here, the a term refers to a measure of the angular momentum of the black hole. In particular, this solution corresponds to a black hole with angular momentum $J = Ma$ where M is the associated mass and $0 \leq a \leq M$. If $a = J = 0$ then this solution reduces to the Schwarzschild static solution.

2.3 Weyl Solutions

2.3.1 Weyl Solutions in 4 Dimensions

Weyl [28] famously determined the general static axisymmetric solution of the vacuum Einstein equations [6]. The metric in this case took the form

$$ds^2 = -e^{2U}dt^2 + e^{-2U}(r^2 d\phi^2 + e^{2\gamma}(dr^2 + dz^2)). \tag{2.14}$$

In this formulation $U(r, z)$ is an axisymmetric solution of Laplace's equation in a 3 dimensional flat spacetime [6]. In this way t is a time coordinate and $\frac{d}{dt}$ is a Killing vector field, whereas ϕ is an angular coordinate with period 2π and $\frac{d}{d\phi}$ is a Killing vector field that generates the axisymmetry. Further $r > 0$ and $-\infty < z < \infty$ can be thought of as cylindrical coordinates in R^3. By introducing the unphysical coordinate θ, the metric can be written as

$$ds^2 = r^2 d\theta^2 + dr^2 + dz^2 \tag{2.15}$$

such that γ satisfies

$$\frac{\partial \gamma}{\partial r} = r\left[\left(\frac{\partial U}{\partial r}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right] \tag{2.16}$$

and

$$\frac{\partial \gamma}{\partial z} = 2r\frac{\partial U}{\partial r}\frac{\partial U}{\partial z}. \tag{2.17}$$

U_i is independent of the unphysical coordinate θ. These conditions on the partial derivatives of γ come from the Einstein equations for this class of geometry [7]. In this case γ must be integrable. We can confirm this fact by taking the difference of the second partial derivatives

$$\frac{\partial^2\gamma}{\partial z\partial r} = 2r\left[\left(\frac{\partial U}{\partial r}\right)\left(\frac{\partial^2 U}{\partial z\partial r}\right) - \left(\frac{\partial U}{\partial z}\right)\left(\frac{\partial^2 U}{\partial^2 z}\right)\right] \tag{2.18}$$

and

$$\frac{\partial^2\gamma}{\partial r\partial z} = 2\frac{\partial U}{\partial r}\frac{\partial U}{\partial z} + 2r\left(\frac{\partial^2 U}{\partial^2 r}\frac{\partial U}{\partial z} + \frac{\partial U}{\partial r}\frac{\partial^2 U}{\partial r\partial z}\right). \tag{2.19}$$

By taking the difference of these two terms we get that

$$\frac{\partial^2\gamma}{\partial r\partial z} - \frac{\partial^2\gamma}{\partial z\partial r} = 2r\frac{\partial U}{\partial z}\left(\frac{\partial^2 U}{\partial^2 z} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial^2 r}\right) \tag{2.20}$$

which is identically zero when U is a solution to Laplace's equation. Specifically

$$\frac{\partial^2 U}{\partial^2 z} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial^2 r} = 0 \tag{2.21}$$

Hence we know that the mixed second partial derivatives of γ are equal, thus γ must be integrable.

2.3.2 Weyl Spacetimes in Higher Dimensions

Emparan and Reall's 2002 work [6] sought to generalize this result to higher dimensional spacetimes. In particular, we are interested in the case of 5 dimensions.

In this work, Emparan and Reall obtain their desired results. Specficially, we begin with a metric of the canonical form:

$$ds^2 = \sum_{i=0}^{D-3}\epsilon_i e^{2U_i}dx_i^2 + e^{2\nu}(dr^2 + dz^2). \tag{2.22}$$

where x_i are directions corresponding to the Killing vectors and $\epsilon_i = \pm 1$. In this initial case we will choose $\epsilon_0 = -1$ and $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{D-3} = 1$.

The vacuum Einstein field equations read that $R_{ab} = 0$. From Emparan and Reall

[6], by considering the 'i' and 'j' components yields the result

$$\sum_{i=0}^{D-3} U_i = \log(w_1(z_1) + w_2(z_2)). \tag{2.23}$$

When z_1 and z_2 are complex conjugates then so are w_1 and w_2 but w_1 and w_2 are independent real functions if z_1 and z_2 are real. Here, z_1 and z_2 are auxilliary coordinates where they are complex conjugates if the space is spacelike, and indpendent real coordinates if the space is timelike. By the introduction of real coordinates (r, z) to be consistent with the format we have been using, namely $w_1 = r + iz$ and $w_2 = r - iz$, we can rewrite this equation as

$$\sum_{i=0}^{D-3} U_i = \log(r) + Const \tag{2.24}$$

where for the purposes of our work this constant will be taken to be $Const = 0$. As well, each U_i is an axisymmetric solution to the 3 dimensional Laplace's equation [11]

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right) U_i = 0. \tag{2.25}$$

Due to Einstein's equations we also have the constraints [7, 11] that

$$\partial_r \nu = \frac{-1}{2r} + \frac{r}{2}\sum (\partial_r U_i)^2 - (\partial_z U_i)^2 \tag{2.26}$$

$$\partial_z \nu = r \sum \partial_r U_i \partial_z U_i. \tag{2.27}$$

The integrability conditions on ν are $\partial_z \partial_r \nu = \partial_r \partial_z \nu$. We know that these conditions are satisfied provided U_i satisfy the axisymmetric Laplace's equation in R^3 as in the 4 dimensional case. This follows a similar pattern as the previous study of the integrability conditions for γ. It can be easily determined that this construction is consistent with ν being integrable.

These conditions are a simplified way of expressing the results from Emparan and Reall's earlier work [6]. With these conditions in place, any solution to the metric (2.22) following conditions (2.25-2.27) are Generalized Weyl Solutions [11]. This is to say that these are the necessary and sufficient conditions to determine a solution. This forms the basis for the design of our work moving forward.

2.4 Rod Diagrams and Minkowski Space

In this work we will use so-called "Rod Diagrams" in order to illustrate the behaviour and characteristics of the solution to a particular system. These diagrams are commonly used in discussing Weyl solutions and are used throughout major publications on the subject. In particular, we will follow the style as outlined by Emparan and Reall [6].

These rods are constructed from the combination of the various solutions to the Laplace's equation $\nabla^2 U = 0$, given previously in (2.21). In our case we break the function U into a combination of functions, denoted by U_i. Most solutions of interest (e.g. Schwarzschild) have U_i that correspond to the Newtonian potentials produced by finite and infinite thing rods lying along the z-axis. In particular the sum of the $U_i = \log r$, and $\log r$ is the Newtonian potential corresponding to an infinitely long rod on the z-axis. This is derived in (2.24). This solution is axisymmetric in that it does not depend on angular orientation.

We can study an example of a simple system to gain a deeper understanding of this tool. One such example to be used is the study of Minkowski Spacetime. It is known in general that we can write all metrics in their canonical form (2.22).

The typical form for the Minkowski metric is given by

$$ds^2 = -dt^2 + dR^2 + R^2 \left(d\theta^2 + \sin(\theta)^2 d\phi_1^2 + \cos(\theta)^2 d\phi_2^2 \right) \tag{2.28}$$

however, this form is not as useful as the canonical form (2.22). Thus, we apply coordinate transformations of the form

$$r = \frac{1}{2}R^2\sin(2\theta) \tag{2.29}$$

$$z = \frac{1}{2}R^2\cos(2\theta) \tag{2.30}$$

$$R^2 = 2\sqrt{r^2+z^2}. \tag{2.31}$$

There are three major cases corresponding to this construction, however two of these reduce rather simply to trivial solutions. We will instead study a particular case of 5 dimensional Minkowski spacetime. In this case we have two semi-infinite rods, each extending from some shared source point, with U_0 being constant (in particular $U_0 = 0$). Here, it is enough to say that corresponds to the potential of a semi-infinite rod extending from $z = a$ to ∞ and $r = 0$ while U_2 corresponds to the potential of a semi-infinite rod extending from $z = a$ to $-\infty$ and $r = 0$. Without loss of generality we may simply choose $a = 0$, however for the purposes of illustration we will leave this as a constant a. Figure 2.1 is the rod diagram that corresponds to this construction.

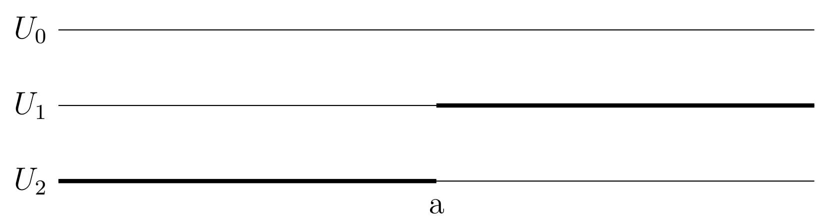

Figure 2.1: Rod diagram for Minkowski

The two semi infinite rods correspond in the (R,θ) coordinates to $\theta = 0$ and $\theta = \frac{\pi}{2}$ where $\frac{d}{d\phi_1}$ and $\frac{d}{d\phi_2}$ vanish. The two semi infinite rods meet at a single point, $R = 0$, where both $\frac{d}{d\phi_1}$ and $\frac{d}{d\phi_2}$ vanish.

According to Emparan and Reall [6] there is a standard construction for the solution of semi-infinte rods. This takes the form of

$$U = \rho\log(\mu_k) \tag{2.32}$$

for semi-infinite rods on the positive axis where ρ is the linear density. In our case, $\rho = \frac{1}{2}$ [7]. For semi-infinite rods on the negative axis

$$U = \rho \log \left(\frac{r^2}{\mu_k} \right). \tag{2.33}$$

Here, we are using the shorthand that

$$\mu_k = \sqrt{r^2 + (z - a_k)^2} - (z - a_k). \tag{2.34}$$

In this notation, the a_k terms represent the “node” points where our rods meet. In our case there is only one node and it is a constant $a_k = a$.

In this case it is clear to see that

$$U_1 = \frac{1}{2} \log \left(\sqrt{r^2 + (z - a)^2} - (z - a) \right) \tag{2.35}$$

$$U_2 = \frac{1}{2} \log \left(\frac{r^2}{\sqrt{r^2 + (z - a)^2} - (z - a)} \right). \tag{2.36}$$

In addition, we know that

$$\sum_{i=0}^{D-3} U_i = \log(r). \tag{2.37}$$

Thus, we can see

$$\sum_{i=0}^{D-3} U_i = U_0 + U_1 + U_2 = U_0 + \log(r). \tag{2.38}$$

Hence, we have the result that

$$U_0 = 0 \tag{2.39}$$

$$U_1 = \frac{1}{2}\log\left(\sqrt{r^2+(z-a)^2}-(z-a)\right) \tag{2.40}$$

$$U_2 = \frac{1}{2}\log\left(\frac{r^2}{\sqrt{r^2+(z-a)^2}-(z-a)}\right). \tag{2.41}$$

Thanks to our coordinate transform, we can compare the metrics (2.22) and (2.28). For simplicity we will let $a = 0$. Using the relationships between r, z, R, and θ we can obtain the result that

$$e^{2\nu}(dr^2 + dz^2) = dR^2 + R^2 d\theta^2 \tag{2.42}$$

and thus

$$e^{2\nu} = \frac{1}{R^2} = \frac{1}{2\sqrt{r^2+z^2}}. \tag{2.43}$$

Thus the general form for our metric is

$$\begin{aligned} ds^2 = -dt^2 + \left(\frac{r^2}{\sqrt{r^2+(z-a)^2}-(z-a)}\right) d\phi_1^2 + \left(\sqrt{r^2+(z-a)^2}-(z-a)\right) d\phi_2^2 \\ + \frac{1}{2\sqrt{r^2+z^2}}\left(dr^2+dz^2\right). \end{aligned} \tag{2.44}$$

2.5 Example: A spacetime containing two 2-cycles

In this study, we will consider the case of the line element form of the metric

$$ds^2 = -e^{2U_0}dt^2 + \sum_{\alpha=1}^{D-3} e^{2U_\alpha}(d\phi^\alpha)^2 + e^{2\nu}(dr^2+dz^2). \tag{2.45}$$

In this case, we assume that the ϕ^α are periodic angular coordinates, $r > 0$ is a radial coordinate and $z \in \mathbf{R}$. When $r = 0$ this corresponds to the axial symmetry or where a horizon is.

We will demand that the angular components of the metric go to 0 under conditions to be determined. In particular, we will demand that the radial component r goes to 0. As a comment, we know that in all cases of this type there must be an even number of bubbles. This is because the Weyl class requires all Killing vectors be orthogonal so there can be no “mixing“ of directions and the metric is orthogonal.

Following from Emparan and Reall [6], we know that we can construct the functions U_1 and U_2 by requiring that they solve the Laplace Equation and based on the regions where we want the e^{U_α} functions to be 0, or vanish. These functions U_α are functions of r and z, $U_\alpha = U_\alpha(r, z)$.

At this moment, we will consider the case where e^{U_1} is 0 which occurs when $r = 0$ and -$\infty < z < c$ or $b < z < a$. That is a semi-infinite rod and a finite rod. Similarly, we impose that e^{U_2} be 0 when $r = 0$ and $c < z < b$ or $a < z < +\infty$. By the properties of the metric, we can shift the undefined terms above so that $b = 0$. Additionally, we can scale the values so that $c = -1$. Refer to Figure 2.2 for the final design of the regions defined.

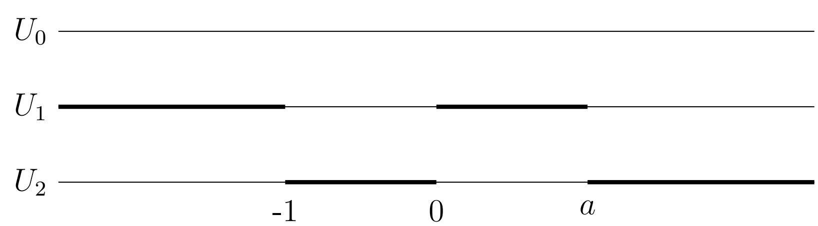

Figure 2.2: Rod diagram corresponding to two 2-cycles or bubbles

In this case, a is a parameter which we will leave intentionally undefined so that we may impose restrictions on it later if needed. The finite rods, such as for U_1 from 0 to a, correspond to “bubbles” and hence our interest in constructing such a model.

Now, according to Emparan and Reall, we can construct the functions U_α so that these conditions are met. As discussed in the case of Minkowski Spacetime, for positive semi-infinite rods

$$U = \rho \log(\mu_k) \tag{2.46}$$

where ρ is the linear density. Once again, in our case, $\rho = \frac{1}{2}$ [7]. For negative semi-infinite rods

$$U = \rho \log\left(\frac{r^2}{\mu_k}\right). \tag{2.47}$$

Finally, for finite rods,

$$U = \rho \log\left(\frac{\mu_{k-1}}{\mu_k}\right). \tag{2.48}$$

In these cases

$$\mu_k = \sqrt{r^2 + (z - a_k)^2} - (z - a_k) \tag{2.49}$$

where the a_k are the node points, such as those indicated on the above rod diagram. Thus, for the case of U_1, we have

$$U_1 = \frac{1}{2} \log\left(\frac{r^2}{\mu_1}\frac{\mu_2}{\mu_3}\right). \tag{2.50}$$

This is due to the combination of the finite rod and the negative semi-infinite rod. Similarly, we can see that

$$U_2 = \frac{1}{2} \log\left(\mu_3 \frac{\mu_1}{\mu_2}\right). \tag{2.51}$$

Finally, we know that

$$\sum_{i=0}^{D-3} U_i = \log(r). \tag{2.52}$$

Thus, we have

$$\sum_{i=0}^{D-3} U_i = U_0 + U_1 + U_2 = \log(r), \tag{2.53}$$

$$U_0 + \frac{1}{2} \log\left(\frac{r^2}{\mu_1}\frac{\mu_2}{\mu_3}\right) + \frac{1}{2} \log\left(\mu_3 \frac{\mu_1}{\mu_2}\right) = U_0 + \log(r). \tag{2.54}$$

Hence, $U_0 = 0$ in this case. Thus the final form of our metric can be written as

$$ds^2 = -dt^2 + \frac{r^2}{\mu_1}\frac{\mu_2}{\mu_3}(d\phi^1)^2 + \mu_3\frac{\mu_1}{\mu_2}(d\phi^2)^2 + e^{2\nu}(dr^2 + dz^2). \tag{2.55}$$

where determining ν is a more complicated process. This will be discussed further in the next chapter.

2.6 Conical Singularities

The properties of Weyl solutions are encoded in the rod diagrams as explained above. On such rods, either spatial Killing vectors degenerate (these are symmetry axes) or a timelike Killing field becomes null (these are horizons). For the spacetime to be smooth, a spatial Killing field must degenerate smoothly so that, near the axis, the spacetime is diffeormorphic to flat space and not just a 'wedge' of it. Otherwise, we say a 'conical singularity' forms . Intuitively this can be thought of as cutting out a 'wedge' of the two-dimensional plane with vertex at the origin and then identifying the edges - this would leave a 'cone'.

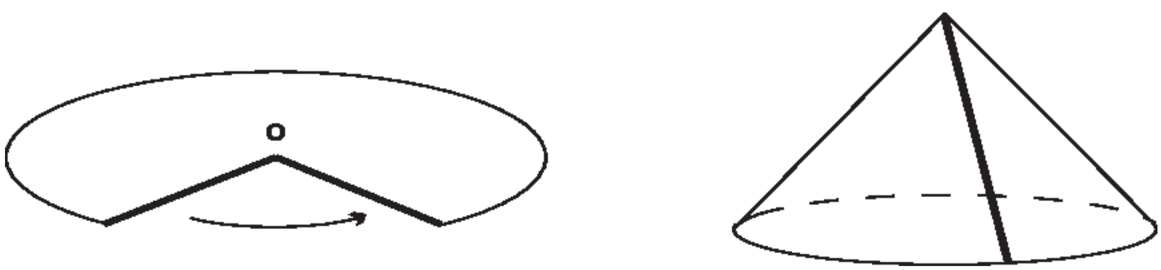

Figure 2.3: This figure demonstrates how a conical singularity may be visualized in an intuitive way [14].

In order to remove such coordinate singularities, we must fix the period of an angle ϕ corresponding to the Killing field $\frac{\partial}{\partial\phi}$ to be $\Delta\eta$, which is given by

$$\Delta\eta = 2\pi \lim_{r\to 0} \sqrt{\frac{r^2 e^{2\nu}}{g_{ij}v^i v^j}} \tag{2.56}$$

where g_{ij} is our metric and the $v^i \frac{\partial}{\partial \phi_i}$ denotes the Killing vector field which is vanishing on the rod. The v^i must be integers so that the Killing vector fields $v^i \frac{\partial}{\partial \phi_i}$ have closed orbits with period 2π. Here, we have used the notation consistent with Harmark [11]. In general, an arbitrarily linear combination of $\frac{\partial}{\partial \phi^1}$, $\frac{\partial}{\partial \phi^2}$ can vanish along a rod. Additionally, ν is determined using a prescribed technique to be outlined in the following section. In the cases we will study, we have no off-diagonal terms to be included. Additionally, each conical singularity corresponds to a particular potential U_i. Thus the $\Delta\eta$ term belonging to U_1 will not be the same as term corresponding to U_2.

In examples which are asymptotically flat spacetimes, we will require that the period of the orbit of the Killing field be $\Delta\eta = 2\pi$. Hence we expect that the limit term will approach 1 as $r \to 0$. In each of these cases, we will have to consider the values for the z term as well. In our discussion of the rod diagrams we know that our potential term lives on the z axis. Thus we have $z \in (-\infty, \infty)$ and we must study the cases that correspond to the potential we are analyzing. For example, in the simple Minkowski Spacetime construction we first saw, we know that if we are considering the potential U_1 we must say that $z \in [0, \infty)$

In the case of Minkowski Spacetime, we have determined that $e^{2\nu} = \frac{1}{R^2}$. By considering (2.56) in the case of U_1, we may see the machnics of the limit. In this case (2.56) becomes

$$\begin{aligned} \Delta\eta &= 2\pi \lim_{r \to 0} \sqrt{\frac{r^2 R^{-2}}{e^{2U_1}}} \\ &= 2\pi \lim_{r \to 0} \sqrt{\frac{r^2 R^{-2}}{\sqrt{r^2 + z^2} - z}} \end{aligned} \tag{2.57}$$

By conducting a series expansion and allowing $r \to 0$ we obtain the result that $\Delta\eta = 2\pi$. Thus we confirm that Minkowski spacetime is asymptotically flat.

In the coming sections we will study examples with more complicated conical deficits. We desire to obtain systems which are asymptotically flat. We will study the restrictions under which enforce this result. We know that systems which are asymptotically flat have nice properties and are well understood, so enforcing this condition is particularly useful. In some of these cases we will have to determine whether it is possible for the system to be made asymptotically flat without reducing the system to being trivial or “breaking” the construction.

Chapter 3

Methodology

In the previous chapter, we demonstrated how one could build a solution of the static vacuum Einstein equations by choosing a rod structure corresponding to solutions of Laplace's equation in R^3. To find an explicit solution one would like to integrate for the function ν. This equation is integrable but difficult to integrate explicitly. We now outline a method for performing the integration. This will be necessary for any further analysis of these systems.

3.1 Methodology for solving PDEs

In this section we will briefly discuss our established method for solving particular PDEs which we will see in upcoming work.

From Iguchi and Mishima [12], given a particular structure for $\bar{U}_c$ we know that the following PDEs for the unknown function γ_{cd}.

$$\partial_r \gamma_{cd} = r[\partial_r \bar{U}_c \partial_r \bar{U}_d - \partial_z \bar{U}_c \partial_z \bar{U}_d] \tag{3.1}$$

$$\partial_z \gamma_{cd} = r[\partial_r \bar{U}_c \partial_z \bar{U}_d + \partial_z \bar{U}_c \partial_r \bar{U}_d] \tag{3.2}$$

$$\bar{U}_c = \frac{1}{2}\log[R_c + (z-c)] \tag{3.3}$$

have solution $\gamma_{cd} = \frac{1}{2}[\bar{U}_c + \bar{U}_d - \frac{1}{2}\log Y_{cd}]$ where $Y_{cd} = R_c R_d + (z-c)(z-d) + r^2$ and $R_c = \sqrt{r^2 + (z-c)^2}$.

Thus, if our system of PDEs has this structure, we know that the solution exists and its form is γ_{cd}.

In our case, we know that our system of coupled PDEs in general looks like

$$\partial_r \nu = \frac{-1}{2r} + \frac{r}{2}\sum(\partial_r U_i)^2 - (\partial_z U_i)^2 \tag{3.4}$$

$$\partial_z \nu = r\sum \partial_r U_i \partial_z U_i. \tag{3.5}$$

We can see an obvious parallel between this system and the above conditions in the form of the derivatives being split between r and z partial derivatives and with general structure of squared derivative for the r derivative and mixed derivatives for the z derivative.

If we rewrite the U functions as $\bar{U}$ functions, then replacing them appropriately, we return the $\partial_r \nu$ equation in terms of only $\bar{U}$ as we desire. Additionally, due to the constraint that $\sum U_i = \log r$ we will cancel the $\frac{1}{2r}$ term entirely. This leaves us a sum of only $\bar{U}$ functions.

In this form, the equation is a sum of mixed partial derivatives. By inspection, one can "pick out" the appropriate forms of $\partial_r \gamma_{cd}$ to substitute. Once the equation is written in this form, one simply has a sum of partial derivatives with respect to r. Pulling through an integration constant, we obtain a "guess" at the solution.

Similarly, we perform the same operation on the $\partial_z \nu$ equation, and achieve the same result, namely that our new coupled equations take on the form demanded in Iguchi and Mishima [12]. It is expected that the partial derivative $\partial_z \nu$ will result in the same "guess" down to the constant of integration. Thus, we will know the form ν must take on to solve this system.

Hence, our system of PDEs is solved, ν is determined to a constant of integration (which may be determined via asymptotics), and our metric is completely determined.

3.2 Determining ν

3.2.1 Sample Case - Schwarzschild

Given the requirements from Reall and Emparan [7] we have that

$$\partial_r \nu = \frac{-1}{2r} + \frac{r}{2} \sum (\partial_r U_i)^2 - (\partial_z U_i)^2 \tag{3.6}$$

$$\partial_z \nu = r \sum \partial_r U_i \partial_z U_i \tag{3.7}$$

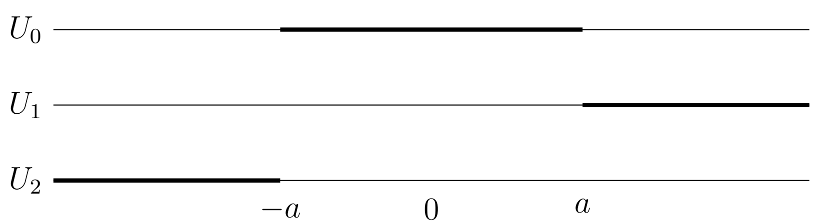

Figure 3.1: Rod diagram for 5 Dimensional Schwarzschild Black Hole

In the Schwarzschild case the basic solutions are well understood

$$U_2 = \frac{1}{2} \log[\sqrt{(z+a)^2 + r^2} + (z+a)] \tag{3.8}$$

$$U_1 = \frac{1}{2} \log[\sqrt{(z-a)^2 + r^2} - (z-a)] \tag{3.9}$$

$$U_0 = \log r - U_2 - U_1. \tag{3.10}$$

Based on the methodology outlined in Iguchi and Mishima [12] we denote

$$
\begin{aligned}
U_2 &= \bar{U}_2 && (3.11)\\
\bar{U}_1 &= \frac{1}{2}\log[\sqrt{(z-a)^2+r^2}+(z-a)] && (3.12)\\
U_1 &= \log r - \bar{U}_1 && (3.13)\\
U_0 &= \log r - U_2 - U_1 = \bar{U}_2 - \bar{U}_1. && (3.14)
\end{aligned}
$$

Expanding out the previously mentioned requirements for the ν functions we have

$$
\begin{aligned}
\partial_r\nu &= \frac{-1}{2r} + \frac{r}{2}\sum(\partial_r U_i)^2 - (\partial_z U_i)^2\\
&= \frac{-1}{2r} + \frac{r}{2}[(\partial_r U_0)^2 - (\partial_z U_0)^2 + (\partial_r U_1)^2 - (\partial_z U_1)^2 + (\partial_r U_2)^2 - (\partial_z U_2)^2]
\end{aligned}
\tag{3.15}
$$

then, substituting U_0, U_1, and U_2 as above yields,

$$
\begin{aligned}
\partial_r\nu = \frac{-1}{2r} &+ \frac{r}{2}[(\partial_r\bar{U}_1 - \bar{U}_2)^2 - (\partial_z\bar{U}_1 - \bar{U}_2)^2 + (\partial_r \log r - \bar{U}_1)^2 -\\
&\qquad (\partial_z \log r - \bar{U}_1)^2 + (\partial_r\bar{U}_2)^2 - (\partial_z\bar{U}_2)^2]\\
= \frac{-1}{2r} &+ \frac{r}{2}[\frac{1}{r^2} - \frac{2\partial_r\bar{U}_1}{r} + 2(\partial_r\bar{U}_1)^2 + 2(\partial_r\bar{U}_2)^2 - 2(\partial_z\bar{U}_1)^2 -\\
&\qquad 2(\partial_z\bar{U}_2)^2 - 2\partial_r\bar{U}_1\partial_z\bar{U}_2 - 2\partial_z\bar{U}_1\partial_r\bar{U}_2]\\
= -\partial_r\bar{U}_1 &+ r[(\partial_r\bar{U}_1)^2 + (\partial_r\bar{U}_2)^2 - (\partial_z\bar{U}_1)^2 - (\partial_z\bar{U}_2)^2 -\\
&\qquad \partial_r\bar{U}_1\partial_z\bar{U}_2 - \partial_z\bar{U}_1\partial_r\bar{U}_2].
\end{aligned}
\tag{3.16}
$$

Similarly, we will study the first partial z-derivative,

$$
\begin{aligned}
\partial_z\nu &= r[\partial_r U_0\partial_z U_0 + \partial_r U_1\partial_z U_1 + \partial_r U_2\partial_z U_2]\\
&= r[\partial_r(\bar{U}_1 - \bar{U}_2)\partial_z(\bar{U}_1 - \bar{U}_2) + \partial_r(\log r - \bar{U}_1)\partial_z(\log r - \bar{U}_1) + \partial_r\bar{U}_2\partial_z\bar{U}_2]\\
&= -\partial_z\bar{U}_1 + r[2\partial_r\bar{U}_1\partial_z\bar{U}_1 + 2\partial_r\bar{U}_2\partial_z\bar{U}_2 - \partial_r\bar{U}_1\partial_z\bar{U}_2 - \partial_z\bar{U}_1\partial_r\bar{U}_2].
\end{aligned}
\tag{3.17}
$$

In Iguchi and Mishima [12], a certain class of solution is defined. As discussed previously, these solutions will act as our initial guess to possible solutions for our coupled differential equations. These functions are given as follows:

$$
\begin{aligned}
\gamma_{cd} &= \frac{1}{2}\bar{U}_c + \frac{1}{2}\bar{U}_d - \frac{1}{4}\log Y_{cd} \\
Y_{cd} &= R_c R_d + (z-c)(z-d) + r^2 \\
R_c &= \sqrt{r^2 + (z-c)^2}.
\end{aligned}
\tag{3.18}
$$

In this case, the functions γ_{cd} satisfy the coupled differential equations (3.1) and (3.2)

Using these above restrictions, we will rewrite the given version of our partial derivatives in terms of these functions.

$$
\begin{aligned}
\partial_z(\nu + \bar{U}_1) &= r[2\partial_r\bar{U}_1\partial_z\bar{U}_1 + 2\partial_r\bar{U}_2\partial_z\bar{U}_2 - \partial_r\bar{U}_1\partial_z\bar{U}_2 - \partial_z\bar{U}_1\partial_r\bar{U}_2] \\
&= \partial_z\gamma_{11} + \partial_z\gamma_{22} - \partial_z\gamma_{12} \\
\nu + \bar{U}_1 &= \gamma_{11} + \gamma_{22} - \gamma_{12} + C.
\end{aligned}
\tag{3.19}
$$

When we consider the r derivative we see,

$$
\begin{aligned}
\partial_r(\nu + \bar{U}_1) &= r[(\partial_r\bar{U}_1)^2 + (\partial_r\bar{U}_2)^2 - (\partial_z\bar{U}_1)^2 - (\partial_z\bar{U}_2)^2 - \partial_r\bar{U}_1\partial_z\bar{U}_2 - \partial_z\bar{U}_1\partial_r\bar{U}_2] \\
&= \partial_r\gamma_{11} + \partial_r\gamma_{22} - \partial_r\gamma_{12} \\
\nu + \bar{U}_1 &= \gamma_{11} + \gamma_{22} - \gamma_{12} + C.
\end{aligned}
\tag{3.20}
$$

Thus, we have determined that $\nu = \gamma_{11} + \gamma_{22} - \gamma_{12} - \bar{U}_1 + C$ where the γ's are as defined above and C is a constant of integration which we will now determine.

In order to determine the integration constant we need to consider the asymptotics of the solution. We have chosen a simple example here and expect that the asymptotic infinity yields a flat spacetime. To see this, we consider a series expansion of $e^{2\nu}$. In this case, as $R \to \infty$ we obtain that

$$
e^{2\nu} = \frac{\sqrt{2}e^{2C}}{R^2} + \mathcal{O}\left(\frac{1}{R^4}\right). \tag{3.21}
$$

Here, we have used the coordinate transformations (2.29-2.31)

It is known that the asymptotic limit of $e^{2\nu}$ in flat space is $\approx \frac{1}{R^2}$. Thus, we see that $e^{2C} = \frac{1}{\sqrt{2}}$. Hence, $C = \frac{1}{4}\log(\frac{1}{2})$. Hence, our final solution to the Schwarzschild metric sample case is

$$\nu = \gamma_{11} + \gamma_{22} - \gamma_{12} - \bar{U}_1 + \frac{1}{4}\log(\frac{1}{2}). \tag{3.22}$$

3.2.2 5 Dimensional spacetime with two 2-cycles

We will construct an asymptotically flat spacetime with no horizon and 2 bubbles. Thanks to the above example we have a rubric to guide us to solving our particular spacetime's ν. After this is determined we will have a complete form of the metric and can proceed onto further analysis.

First, we will consider the following general rod structure diagram,

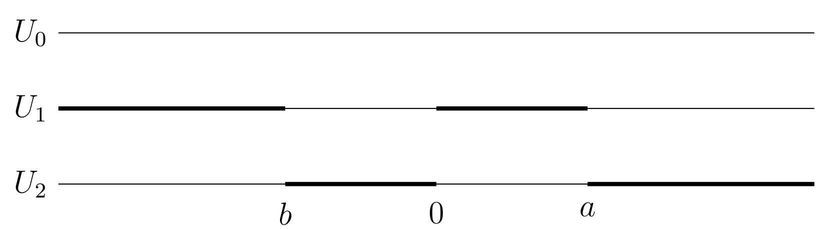

Figure 3.2: Rod diagram for our imposed restrictions with general end points. Here the finite rods along the ϕ_1 directions $(0 < z < a)$ and ϕ_2 $(b < z < 0)$ correspond to two S^2 'bubbles' parameterized by (z, ϕ_2) and (z, ϕ_1) respectively.

In this case, we know that we require the functions $e^{2U_i} = 0$ on the corresponding regions. For example, $e^{2U_1} = 0$ on the intervals from $-\infty$ to b and 0 to a. Using the previously defined functions $\mu_k = \sqrt{r^2 + (z-k)^2} - (z-k)$ and $\bar{\mu}_k = \sqrt{r^2 + (z-k)^2} + (z-k)$ we can see that

$$\begin{aligned} e^{2U_1} &= \frac{r^2\mu_0}{\mu_a\mu_b} = \frac{\bar{\mu}_a\bar{\mu}_b}{\bar{\mu}_0} \\ e^{2U_2} &= \frac{\mu_a\mu_b}{\mu_0} = \frac{r^2\bar{\mu}_0}{\bar{\mu}_a\bar{\mu}_b} \end{aligned} \tag{3.23}$$

give us our desired behaviour. We are using the notation that

$$\bar{U}_a = \frac{1}{2}\log \bar{\mu}_a \tag{3.24}$$

These functions in this notation effectively "turn on" and "turn off" the functions e^{2U_i} starting at each value for a in the positive infinite direction. Rearranging these equations yields

$$\begin{aligned} U_1 &= \bar{U}_a + \bar{U}_b - \bar{U}_0 \\ U_2 &= \log r + \bar{U}_0 - \bar{U}_a - \bar{U}_b. \end{aligned} \tag{3.25}$$

Thanks to the previously established relationship $\sum U_i = \log r$, we can say that $U_0 = 0$. This was previously established in an earlier section via an alternate methodology. Once again, we begin our analysis with the coupled equations (3.4, 3.5). Starting with the ∂_r derivative, we have

$$\begin{aligned} \partial_r \nu &= \frac{-1}{2r} + \frac{r}{2}[(\partial_r U_1)^2 + (\partial_r U_2)^2 - (\partial_z U_1)^2 - (\partial_z U_2)^2] \\ &= \frac{-1}{2r} + \frac{r}{2}[(\partial_r \bar{U}_a + \partial_r \bar{U}_b - \partial_r \bar{U}_0)^2 + (\frac{1}{r} + \partial_r \bar{U}_0 - \partial_r \bar{U}_a - \partial_r \bar{U}_b)^2 \\ &- (\partial_z \bar{U}_a + \partial_z \bar{U}_b - \partial_z \bar{U}_0)^2 - (\partial_z \bar{U}_0 - \partial_z \bar{U}_a - \partial_z \bar{U}_b)^2] \\ &= r[\partial_r \bar{U}_a^2 - \partial_z \bar{U}_a^2 + \partial_r \bar{U}_0^2 - \partial_z \bar{U}_0^2 + \partial_r \bar{U}_b^2 - \partial_z \bar{U}_b^2 \\ &+ 2\partial_r \bar{U}_a \partial_r \bar{U}_b - 2\partial_z \bar{U}_a \partial_z \bar{U}_b - 2\partial_r \bar{U}_0 \partial_r \bar{U}_b + 2\partial_z \bar{U}_0 \partial_z \bar{U}_b \\ &- 2\partial_r \bar{U}_a \partial_r \bar{U}_0 + 2\partial_z \bar{U}_a \partial_z \bar{U}_0] + \partial_r \bar{U}_0 - \partial_r \bar{U}_a - \partial_r \bar{U}_b \\ &= -\partial_r U_1 + \partial_r \gamma_{aa} + \partial_r \gamma_{00} + \partial_r \gamma_{bb} + 2\partial_r \gamma_{ab} - 2\partial_r \gamma_{0a} - 2\partial_r \gamma_{b0} \\ \partial_r(\nu + U_1) &= \partial_r \gamma_{aa} + \partial_r \gamma_{00} + \partial_r \gamma_{bb} + 2\partial_r \gamma_{ab} - 2\partial_r \gamma_{0a} - 2\partial_r \gamma_{b0} \\ \nu + U_1 &= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} + C \end{aligned} \tag{3.26}$$

So we anticipate our solution to be $\nu = \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} - U_1 + C$. However, we must compare against the other condition required. We will now examine the ∂_z term.

$$
\begin{aligned}
\partial_z \nu &= r[\partial_r U_1 \partial_z U_1 + \partial_r U_2 \partial_z U_2] \\
&= r[(\partial_r \bar{U}_a + \partial_r \bar{U}_b - \partial_r \bar{U}_0)(\partial_z \bar{U}_a + \partial_z \bar{U}_b - \partial_z \bar{U}_0) \\
&+ (\frac{1}{r} + \partial_r \bar{U}_0 - \partial_r \bar{U}_a - \partial_r \bar{U}_b)(\partial_z \bar{U}_0 - \partial_z \bar{U}_a - \partial_z \bar{U}_b)] \\
&= r[2\partial_r \bar{U}_a \partial_z \bar{U}_a + 2\partial_r \bar{U}_b \partial_z \bar{U}_b + 2\partial_r \bar{U}_0 \partial_z \bar{U}_0 \\
& 2\partial_r \bar{U}_a \partial_z \bar{U}_b + 2\partial_r \bar{U}_b \partial_z \bar{U}_a - 2\partial_r \bar{U}_a \partial_z \bar{U}_0 - 2\partial_r \bar{U}_0 \partial_z \bar{U}_a \\
&- 2\partial_r \bar{U}_0 \partial_z \bar{U}_b - 2\partial_r \bar{U}_b \partial_z \bar{U}_0] + \partial_r \bar{U}_0 - \partial_r \bar{U}_a - \partial_r \bar{U}_b \\
&= -\partial_z U_1 + \partial_z \gamma_{aa} + \partial_z \gamma_{00} + \partial_z \gamma_{bb} + 2\partial_z \gamma_{ab} - 2\partial_z \gamma_{0a} - 2\partial_z \gamma_{b0} \\
\partial_z(\nu + U_1) &= \partial_z \gamma_{aa} + \partial_z \gamma_{00} + \partial_z \gamma_{bb} + 2\partial_z \gamma_{ab} - 2\partial_z \gamma_{0a} - 2\partial_z \gamma_{b0} \\
\nu + U_1 &= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} + C
\end{aligned}
\tag{3.27}
$$

Hence, we have confirmed that our solution for the function ν to complete our metric is

$$
\begin{aligned}
\nu &= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} - U_1 + C \\
&= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} + \bar{U}_0 - \bar{U}_a - \bar{U}_b + C.
\end{aligned}
\tag{3.28}
$$

We are now tasked with determining the value for the integration constant C. As we performed before, we will be considering the asymptotic limit and comparing to the flat metric. Once again, we consider the series expansion of $e^{2\nu}$

$$
e^{2\nu} \approx \frac{\sqrt{2}e^{2C}}{R^2} + \mathcal{O}\left(\frac{1}{R^4}\right). \tag{3.29}
$$

Thus, we obtain that $C = \frac{1}{4}\log\frac{1}{2}$ once more. The above γ and $\bar{U}$ functions are known and thus ν is known completely and so is our metric.

3.2.3 Exact form of the Metric

As originally stated in (2.22-2.31) our expression for the metric can be written in the form

$$ds^2 = -e^{2U_0}dt^2 + \sum_{\alpha=1}^{D-3} e^{2U_\alpha}(d\phi^\alpha)^2 + e^{2\nu}(dr^2 + dz^2). \tag{3.30}$$

Previously, we have determined

$$U_0 = 0 \tag{3.31}$$

$$U_1 = \frac{1}{2}\log\left(\frac{r^2}{\mu_1}\frac{\mu_2}{\mu_3}\right) = \frac{1}{2}\log\frac{\bar{\mu}_a\bar{\mu}_b}{\bar{\mu}_0} \tag{3.32}$$

$$U_2 = \frac{1}{2}\log\left(\frac{\mu_3\mu_1}{\mu_2}\right) = \frac{1}{2}\log\left(\frac{r^2}{\bar{\mu}_a}\frac{\bar{\mu}_0}{\bar{\mu}_b}\right) \tag{3.33}$$

$$\nu = \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} - U_1 + \frac{1}{4}\log\frac{1}{2} \tag{3.34}$$

$$= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} - U_1 + \frac{1}{2}\log\frac{1}{\sqrt{2}} \tag{3.35}$$

$$= \gamma_{aa} + \gamma_{00} + \gamma_{bb} + 2\gamma_{ab} - 2\gamma_{0a} - 2\gamma_{b0} + \frac{1}{2}\log\frac{\bar{\mu}_0}{\bar{\mu}_a\bar{\mu}_b\sqrt{2}}. \tag{3.36}$$

This leads to a simplified version of our metric being written as

$$ds^2 = -dt^2 + \frac{\bar{\mu}_a\bar{\mu}_b}{\bar{\mu}_0}(d\phi^1)^2 + \frac{r^2\bar{\mu}_0}{\bar{\mu}_b\bar{\mu}_a}(d\phi^2)^2 + \frac{\bar{\mu}_0}{\bar{\mu}_a\bar{\mu}_b\sqrt{2}}\frac{e^{2\gamma_{aa}}e^{2\gamma_{00}}e^{2\gamma_{bb}}e^{4\gamma_{ab}}}{e^{4\gamma_{0a}}e^{4\gamma_{b0}}}(dr^2 + dz^2) \tag{3.37}$$

$$= -dt^2 + \frac{\bar{\mu}_a\bar{\mu}_b}{\bar{\mu}_0}(d\phi^1)^2 + \frac{r^2\bar{\mu}_0}{\bar{\mu}_b\bar{\mu}_a}(d\phi^2)^2 + \frac{Y_{0b}Y_{a0}}{Y_{ab}\sqrt{2Y_{aa}Y_{00}Y_{bb}}}(dr^2 + dz^2) \tag{3.38}$$

where the Y_{ij}'s work as previously defined.

It is straightforward to show that vacuum, asymptotically flat solitons are ruled out by applying Stokes' theorem and identities for Killing vector fields to show that such solitons must have zero mass and hence must be identically Minkowski spacetime see [17].

3.3 Spacetime containing a Black Hole and Two Bubbles

We will extend our analysis to two cases similar to the one we have most recently studied. Consider finite horizons with time-independent stucture given by the rod diagrams

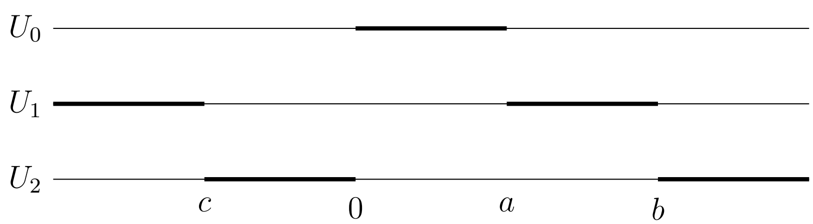

Figure 3.3: Case 1: Horizon with symmetric rod structure. Here the finite rod along $(0 < z < a)$ corresponds to a static black hole horizon

In this case the relevant functions are

$$e^{2U_0} = \frac{\bar{\mu}_a}{\bar{\mu}_0} \tag{3.39}$$

$$e^{2U_1} = \frac{\bar{\mu}_c \bar{\mu}_b}{\bar{\mu}_a} \tag{3.40}$$

$$e^{2U_2} = \frac{r^2 \bar{\mu}_0}{\bar{\mu}_c \bar{\mu}_b} \tag{3.41}$$

$$\begin{aligned} \nu = &- U_0 - U_1 + \gamma_{00} + \gamma_{\mathrm{aa}} + \gamma_{\mathrm{bb}} + \gamma_{\mathrm{cc}} + 2\gamma_{\mathrm{bc}} - \gamma_{\mathrm{ab}} - \gamma_{\mathrm{ac}} - \gamma_{\mathrm{a0}} - \gamma_{\mathrm{b0}} - \gamma_{\mathrm{c0}} \\ &+ \frac{1}{4} \log\left(\frac{1}{2}\right) \end{aligned} \tag{3.42}$$

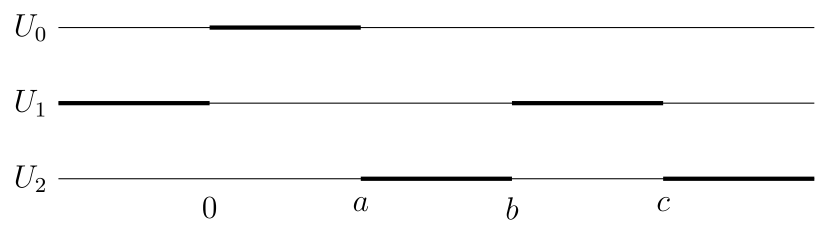

Figure 3.4: Case 2: Horizon with asymmetric rod structure. Here the finite rod along $(0 < z < a)$ corresponds to a static black hole horizon

While in this case the relevant functions are

$$e^{2U_0} = \frac{\bar{\mu}_a}{\bar{\mu}_0} \tag{3.43}$$

$$e^{2U_1} = \frac{\bar{\mu}_0 \bar{\mu}_c}{\bar{\mu}_b} \tag{3.44}$$

$$e^{2U_2} = \frac{r^2 \bar{\mu}_b}{\bar{\mu}_c \bar{\mu}_a} \tag{3.45}$$

$$\begin{aligned}\nu =& -U_0 - U_1 + \gamma_{00} + \gamma_{\rm aa} + \gamma_{\rm bb} + \gamma_{\rm cc} - 2\gamma_{\rm bc} - \gamma_{\rm ab} + \gamma_{\rm ac} - \gamma_{\rm a0} - \gamma_{\rm b0} + \gamma_{\rm c0} \\ &+ \frac{1}{4} \log\left(\frac{1}{2}\right)\end{aligned} \tag{3.46}$$

These two cases will be studied independently as *apriori* we do not know exactly which results may carry over from one case to another.

3.3.1 Conical Singularities with Horizon

Studying the conical singularities of these two cases, we desire to see the impact that the regions, as defined, have on our results. Firstly, we will consider the symmetric case.

In the case of the infinite regions we see, as expected, that the conical singularities are 2π, so we turn our attention to the finite interior regions.

In the case of the function U_1 we have an expression of $\Delta\eta_1 = 2\pi\sqrt{\frac{b(b-a)}{(b-c)^2}}$. Similarly, for the region corresponding to U_2 we see that $\Delta\eta_2 = 2\pi\sqrt{\frac{c(c-a)}{(b-c)^2}}$. We know that $\Delta\eta_1 = \Delta\eta_2 = 2\pi$ in order for the spacetime to be asymptotically flat. Thus, we must check whether this is compatible with regularity on the finite rods (i.e. where e^{2U_1} and e^{2U_2} vanish). As a reminder, here we are assuming that $c < 0 < a < b$.

$$\begin{aligned}
\sqrt{\frac{c(c-a)}{(b-c)^2}} &= \sqrt{\frac{b(b-a)}{(b-c)^2}} \\
c(c-a) &= b(b-a) \\
c^2 - ca &= b^2 - ba \\
b^2 - c^2 &= ba - ca \\
(b-c)(b+c) &= (b-c)a \\
b + c &= a.
\end{aligned} \tag{3.47}$$

Thus, in order for $\Delta\eta_1 = \Delta\eta_2$ it must be the case that $b+c = a$. A necessary condition for $\Delta\eta_1 = \Delta\eta_2 = 2\pi$ is that each of the radical terms need to be identically equal to 1. So we will consider only one of the cases, say U_1.

$$\begin{aligned}
1 &= \sqrt{\frac{b(b-a)}{(b-c)^2}} \\
1 &= \frac{b(b-a)}{(b-c)^2} \\
b^2 - ab + \frac{1}{3}a^2 &= 0
\end{aligned} \tag{3.48}$$

Substituting this into the quadratic equation we find that, with the assumption of $a > 0$

$$\begin{aligned}
b &= \frac{a \pm \sqrt{a^2 - \frac{4}{3}a^2}}{2} \\
&= a\frac{1 \pm \sqrt{\frac{-1}{3}}}{2}
\end{aligned}$$

Hence, we find no real solution to this case where $a > 0$ and only the trivial solution for the case $a = b = c = 0$.

Next we consider the case 2, with asymmetric regions. Once again, we have that the infinite regions give us the expected 2π values. So we will again focus on our trapped finite regions within. First we will examine the case of U_2.

$$
\begin{aligned}
1 &= \sqrt{\frac{b(b-a)}{c(c-a)}} \\
1 &= \frac{b(b-a)}{c(c-a)} \\
b + c &= a.
\end{aligned}
$$

Once again, we see the condition that for U_2 to resolve its conical singularity we have that $b + c = a$. However, in this case we have assumed that $0 < a < b < c$. Thus, it is impossible for this to be met except for the limiting case of $a = b = c = 0$. Now, we examine the U_1 finite region.

$$
\begin{aligned}
1 &= \sqrt{\frac{(c-b)^2}{c(c-a)}} \\
&= \frac{(c-b)^2}{c(c-a)} \\
c^2 - ac &= b^2 - 2bc + c^2 \\
0 &= b^2 - 2bc + ca.
\end{aligned}
$$

Now, solving this using the quadratic equation in terms of some unknown variable b we see

$$
\begin{aligned}
b &= \frac{2c - \sqrt{4c^2 - 4ca}}{2} \\
&= c - \sqrt{c^2 - ca}
\end{aligned}
$$

The $(+)$ term is omitted as we demand that $b < c$. We desire a real solution so we have that

$$c^2 - ca > 0 \rightarrow \quad c^2 > ca \rightarrow \quad c > a$$

Which is true by assumption. However, we also consider the restriction that $c > b > a$. Thus,

$$\begin{aligned} a &< b < c \\ a &< c - \sqrt{c^2 - ca} < c \\ a - c &< -\sqrt{c^2 - ca} < 0 \\ c - a &> \sqrt{c^2 - ca} > 0 \\ (c-a)^2 &> c^2 - ca \\ c^2 + a^2 - 2ac &> c^2 - ac \rightarrow \quad a^2 > ac \rightarrow \quad a > c. \end{aligned}$$

This is a contradiction. In the limiting case where $a = b = c$ we simply obtain the Schwarzschild case that was previously discussed. This shows that there is no way to correct for the conical singularities without breaking down the uniqueness of the system.

In this chapter we have explored the technique we have used to generate solutions to the system of partial differential equations that govern our system. This included an analysis of a simple Schwarzschild case. From there we introduced the particular case we are interested in which is the use of 2-cycles or bubbles. As an initial analysis of these cases we discuss the requirement that the spacetime be regular. Through this lens we study the structure of these cases in order to establish regularity. Thanks to this analysis we determined that there is no way to correct the conical singularities of the problem without the structure of the system reverting to a basic Schwarzschild problem.

In the following section we will introduce the concept of "boosting" as a tool to solve our system and generate solutions. This concept of boosting introduces additional dimensions to our system. By doing this we have the "room" to make transformations to our metric. In doing this we attempt to obtain a new form of our system which can be solved using known methods. In addition, this new form yields the ability to read off more information about the system such as the temperature, surface gravity, or mass.

Chapter 4

Construction of Charged Solutions

4.1 Motivation for Boosting

In Chapter 3 we constructed static, vacuum solutions and showed that they possess conical singularities. In this chapter we will construct charged, static solutions of the Einstein-Maxwell equations. Our goal is to investigate whether the addition of charge could be used to remove the conical singularities of the vacuum solution. In an effort to better understand the 5 dimensional problem, we will employ a solution generating technique used by Emparan and Elvang [5]. In this work, the process outlined takes us from a 5 dimensional spacetime to a 10 dimensional spacetime.

This procedure takes many stages to complete, involving Lorentz Boosts, dualizations, and dimensional reductions. These modifications imbue our system with both momentum and electric charge. These properties we are already familiar with as being key to uniqueness and characterization of black holes in 4 dimensional spacetime.

The basic motivation of these actions are that by performing boosts (coordinate transformations) in the new auxiliary directions, this adds momentum to the solution. Then through a series of duality transformations, these momenta are converted into electric charge after doing dimensional reduction. This yields a charged solution in $D = 5$ instead of a vacuum solution [5].

4.2 Boosting the solution in D=10

In general, we will consider a five-dimensional metric of the form

$$ds_5^2 = -e^{2U_0}dt^2 + e^{2U_1}d\phi_1^2 + e^{2U_2}d\phi_2^2 + e^{2\nu}(dr^2 + dz^2) \tag{4.1}$$

as previously seen in prior calculations. We will extend this definition to a 10 dimensional manifold M_{10} with metric ds_{10}^2 where

$$ds_{10}^2 = ds_5^2 + dy^2 + \sum_{i=4} dy_i^2 \tag{4.2}$$

We have just added 5 flat directions (y, y_1, y_2, y_3, y_4) to our original 5 dimensional metric. Therefore if the original 5 dimensional metric is Ricci flat, then so will the 10 dimensional metric. We will consider the method for boosting and dualizing the metric as outlined in Emparan and Elvang [5], as well as in Myers [22]. Firstly, we define the boosts

$$dt \to \cosh\alpha\ dt + \sinh\alpha\ dy \tag{4.3}$$

$$dy \to \sinh\alpha\ dt + \cosh\alpha\ dy. \tag{4.4}$$

This makes our new metric for the $ds_5^2 + dy^2$ terms

$$\begin{aligned} ds_5^2 + dy^2 \to & -e^{2U_0}(\cosh^2\alpha\ dt^2 + \sinh^2\alpha\ dy^2 + \sinh 2\alpha\ dtdy) + e^{2U_1}d\phi_1^2 \\ & + e^{2U_2}d\phi_2^2 + e^{2\nu}(dr^2 + dz^2) + \cosh^2\alpha\ dy^2 + \sinh^2\alpha\ dt^2 + \sinh 2\alpha\ dtdy \\ & = (\sinh^2\alpha - e^{2U_0}\cosh^2\alpha)dt^2 + e^{2U_1}d\phi_1^2 + e^{2U_2}d\phi_2^2 + e^{2\nu}(dr^2 + dz^2) \\ & + (\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)dy^2 + (1 - e^{2U_0})\sinh 2\alpha\ dtdy. \end{aligned}$$

In supergravity we must incorporate two additional fields, denoted Φ and $B_{\mu\nu}$. Following the methods in Myers [22], and choosing initial duality terms $\Phi = 0$ and $B = 0$, we define the following dualization process. These dualities will occur in the y direction because $\frac{\partial}{\partial y}$ is the associated Killing vector.

This is a T dualization which is a transformation between string and Einstein frame metrics. Initially we begin with the triple $(g_{ab}, \Phi, B) = (g_{ab}, 0, 0)$ in the Einstein frame. Through the application of the T duality we will have new values for these terms, denoted (G_{ab}, Φ', B') in the string frame. Here, g_{ab} and G_{ab} are related by $G_{ab} = e^{\frac{\Phi}{2}} g_{ab}$ where Φ is our dilaton term. If the dilaton is identically zero, $\Phi = 0$, then simply $G_{ab} = g_{ab}$. This transformation preserves the fact that (g_{ab}, Φ, B) being a solution means (G_{ab}, Φ', B') is also a solution. This is not obvious (see [22]). This is a key component to our solution generating technique. Thanks to these processes we can incorporate these underlying fields into the metric. This further complicates our solution but has the potential to provide additional solutions as a result.

$$G_{\mu\nu} = e^{\frac{\Phi}{2}} g_{\mu\nu} \tag{4.5}$$

$$G_{yy} \to \frac{1}{G_{yy}} \tag{4.6}$$

$$e^{2\phi} \to \frac{e^{2\phi}}{G_{yy}} \tag{4.7}$$

$$G_{\mu\nu} \to G_{\mu\nu} - \frac{G_{\mu y} G_{\nu y} - B_{\mu y} B_{\nu y}}{G_{yy}} \tag{4.8}$$

$$G_{\mu y} \to \frac{B_{\mu y}}{G_{yy}} \tag{4.9}$$

$$B_{\mu\nu} \to B_{\mu\nu} - \frac{B_{\mu y} G_{\nu y} - G_{\mu y} B_{\nu y}}{G_{yy}} \tag{4.10}$$

$$B_{\mu y} \to \frac{G_{\mu y}}{G_{yy}} \tag{4.11}$$

where μ and ν do not take the value of y. Now, due to the sparse nature of the metric, many of these terms begin and stay zero, thus we will only discuss those that change or stay non-zero.

$$G_{\mu\nu} = g_{\mu\nu} \tag{4.12}$$

$$G_{yy} \to \frac{1}{\cosh^2\alpha - e^{2U_0}\sinh^2\alpha} \tag{4.13}$$

$$e^{2\phi} \to \frac{1}{\cosh^2\alpha - e^{2U_0}\sinh^2\alpha} \tag{4.14}$$

$$G_{tt} \to G_{tt} - \frac{G_{ty}G_{ty} - B_{ty}B_{ty}}{G_{yy}} = \frac{-e^{2U_0}}{\cosh^2\alpha - e^{2U_0}\sinh^2\alpha} \tag{4.15}$$

$$G_{ty} \to 0 \tag{4.16}$$

$$B_{ty} \to \frac{G_{ty}}{G_{yy}} = \frac{(1-e^{2U_0})\sinh 2\alpha}{2(\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)} \tag{4.17}$$

All other terms not defined above are unchanged. From here, we will consider a second boost. In this case, we use a new boost angle β. We follow the same methodology as outlined above,

$$dt \to \cosh\beta\ dt + \sinh\beta\ dy \tag{4.18}$$

$$dy \to \sinh\beta\ dt + \cosh\beta\ dy. \tag{4.19}$$

Due to the complexity of these metric terms we will omit the specifics of the components when we define the new boosted metric.

$$\begin{aligned}
ds_5^2 + dy^2 \to\ & G_{tt}(\cosh^2\beta\ dt^2 + \sinh^2\beta\ dy^2 + \sinh 2\beta\ dtdy) + G_{\phi_1\phi_1}d\phi_1^2 \\
& + G_{\phi_2\phi_2}d\phi_2^2 + G_{rr}(dr^2 + dz^2) + G_{yy}(\cosh^2\beta\ dy^2 + \sinh^2\beta\ dt^2 + \sinh 2\beta\ dtdy) \\
& = (G_{yy}\sinh^2\beta + G_{tt}\cosh^2\beta)dt^2 + G_{\phi_1\phi_1}d\phi_1^2 + G_{\phi_2\phi_2}d\phi_2^2 + G_{rr}(dr^2 + dz^2) \\
& + (G_{yy}\cosh^2\beta + G_{tt}\sinh^2\beta)dy^2 + (G_{yy} + G_{tt})\sinh 2\beta\ dtdy.
\end{aligned}$$

4.2.1 Proof that B field is Boost Independent

We see above that the metric is affected by the boosts we defined here with the α or β angles. The question is now whether the B field is affected. To determine this, we must define the B field like a metric.

$$B = \frac{B_{ab}}{2!} dx^a \wedge dx^b$$

where the antisymmetric wedge term is defined as $dx^a \wedge dx^b = dx^a \otimes dx^b - dx^b \otimes dx^a$.

From the original definitions of the dualization

$$\begin{aligned} B_{\mu\nu} &\to B_{\mu\nu} - \frac{B_{\mu y} G_{\nu y} - G_{\mu y} B_{\nu y}}{G_{yy}} \\ B_{\mu y} &\to \frac{G_{\mu y}}{G_{yy}} \end{aligned}$$

We initially started with uniform $B_{ab} = 0$. Thus $B_{\mu\nu} \to 0$ excepting for $B_{ty} \to \frac{G_{ty}}{G_{yy}}$. In this case, we now have the term $B_{ty} dt \wedge dy$. So we consider the effect that the boost has on the term $dt \wedge dy$

$$\begin{aligned} dt \wedge dy = dt \otimes dy - dy \otimes dt & \\ &\to (\cosh\beta dt + \sinh\beta) dy \wedge (\sinh\beta + \cosh\beta dy) \\ &\to \cosh\beta dt \wedge \cosh\beta dy + \sinh\beta dy \wedge \sinh\beta dt \\ &\to \cosh\beta dt \wedge \cosh\beta dy - \sinh\beta dt \wedge \sinh\beta dy \\ &\to (\cosh^2\beta - \sinh^2\beta)(dt \wedge dy) \\ &\to dt \wedge dy \end{aligned}$$

Thus the B field is unchanged after boosting. □

4.3 S-duals and T-duals

After the second boost considered above, we must next dualize the results we've obtained. We will follow the process as outlined in Emparan and Elvang [5]. This is schematically defined as the dualizations

$$S \to T \to S \tag{4.20}$$

4.3.1 First S-Dual

Previously we have discussed the meaning and mechanics of T duals. In addition, we may perform so-called, S duals. This is similar to the T dualization process in that performing an S-duality transformation maps a solution to a new solution. In this case the S dual changes the dilaton field, Φ, as well as the fields $C_{\mu\nu}$ and $B_{\mu\nu}$. B is the same field as discussed in the T dualization process, while $C_{\mu\nu}$ is another supergravity field potential, analogous to $B_{\mu\nu}$. During the S dual process the metric is mostly unchanged in that we stay in the same frame (eg string or Einstein) but the metric may be scaled by the new dilaton field.

First, we must define the dilaton involved as

$$e^{\Phi} = \frac{1}{(\cosh^2(\alpha) - e^{2U_0}\sinh^2(\alpha))^{\frac{1}{2}}} \tag{4.21}$$

following from Emparan and Elvang's work [5]. We can also refer to $\frac{1}{(\cosh^2(\alpha) - e^{2U_0}\sinh^2(\alpha))^{\frac{1}{2}}}$ as γ. This is done in anticipation of future manipulations to simplify the notation and calculations. Next our string frame metric is scaled by this term

$$g_{ab} = e^{\frac{-\Phi}{2}} G_{ab} = \gamma^{\frac{-1}{2}} G_{ab} \tag{4.22}$$

Now, we have a new dilaton thanks to the S-dualization. This new dilaton is simply the inverse of the previously defined dilaton.

$$e^{\Phi'} = (\cosh^2(\alpha) - e^{2U_0}\sinh^2(\alpha))^{\frac{1}{2}}. \tag{4.23}$$

This is obviously also just the inverse of the γ term above. Using the methodology as defined in Myers [22], the new string metric is defined as

$$G'_{ab} = e^{\frac{\Phi'}{2}} g_{ab} = \gamma^{-1} G_{ab} = (\cosh^2(\alpha) - e^{2U_0}\sinh^2(\alpha))^{\frac{1}{2}} G_{ab} \tag{4.24}$$

Next we turn our attention to the accompanying fields, $C_{\mu\nu}$ and $B_{\mu\nu}$. $C_{\mu\nu}$ was originally zero, however during the process of S-dualization it takes on non-zero value. These fields exchange values in the following way

$$\begin{pmatrix} B'_{\mu\nu} \\ C'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} C_{\mu\nu} \\ -B_{\mu\nu} \end{pmatrix}. \tag{4.25}$$

The only non-zero B field term is $B_{ty} = \frac{(1-e^{2U_0})\sinh 2\alpha}{2(\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)}$, and $C_{\mu\nu} = 0$ to begin then our new terms are

$$B_{\mu\nu} = 0 \tag{4.26}$$

$$C_{ty} = \frac{(e^{2U_0}-1)\sinh 2\alpha}{2(\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)} \tag{4.27}$$

with all other terms being identically 0. Here we have dropped the "primes" for notation.

4.3.2 T(1234)

Here we will T-dual on each of the four "independent" directions, those denoted y_i for $i = 1, \ldots, 4$. These four dualities will have the same effect on the metric. We will walk through one case, say the y_1 direction, and the other directions will follow suit in the obvious way.

Consider $\mathbf{T(1)}$ for our first method. Using the previously defined steps for dualities we see

$$e^{\phi'} = e^{-\phi} = \gamma \tag{4.28}$$

$$G'_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu 1} G_{\nu 1}}{G_{11}} = G_{\mu\nu} \tag{4.29}$$

$$G'_{\mu 1} = \frac{B_{\mu 1}}{G_{11}} = 0 \tag{4.30}$$

$$B'_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu 1} G_{\nu 1} - G_{\mu 1} B_{\nu 1}}{G_{11}} = 0 \tag{4.31}$$

$$B'_{\mu 1} = \frac{G_{\mu 1}}{G_{11}} = 0 \tag{4.32}$$

$$G'_{11} = \frac{1}{G_{11}} = \gamma \tag{4.33}$$

and all other terms remaining the same. In the future, we will drop the primes as these are the new values for the relevant terms. Hence, we can see that the only changes here are the dilaton and G_{11} terms flip.

Repeating this process for the other three directions, one sees that the dilaton will flip three more times, resulting in it returning to its original value. However, since we undergo the T-dualization process the dilaton must be flipped one more time to yield $e^{-\frac{\phi}{2}} = \gamma^{\frac{1}{2}}$. As well, the other G_{ii} terms will flip on themselves as well. Hence, our full metric is now

$$ds_{10}^2 = \gamma^{-1} ds_6^2 + \gamma \sum_{i=4} dy_i^2 \tag{4.34}$$

as expected. This agrees with the results of Emparan and Elvang [5]. The S duality procedure produced a new type of 2-form field, $C_[2]$. When such a field is present, the T duality transformations will act on this field to produce both higher and lower rank Ramond-Ramond field forms as outlined in Marolf [19]. Schematically, if we perform a T duality in the direction z then if we have field strengths $F_{[n+1]}$ and $F_{[n]}$ then we produce new field strengths which are schematically given by

$$\begin{aligned} F'_{[n],\alpha_1\alpha_2\ldots\alpha_n} &= C_1 F_{[n+1],z\alpha_1\alpha_2\ldots\alpha_n} \\ F'_{[n],z\alpha_1\alpha_2\ldots\alpha_{n-1}} &= C_2 F_{[n-1],\alpha_1\alpha_2\ldots\alpha_{n-1}}. \end{aligned} \tag{4.35}$$

Here the α's represent the various coordinate directions. Equation (4.35) is a way of associating an (n) form potential to an (n+1) form field. In addition, this allows us to relate the higher order forms we need to the known 2-form $C_{\mu\nu}$ via its derivative.

Using this set of rules, we will work our way "up" the chain of expressions. We know that we must have "off diagonal" terms in order for this expression to work in the wedge product basis. In our present solution (after performing the first S duality) the only field strength we have is the 3-form $F_{[3]} = dC_{[2]}$ and so since we only have C_{ty} components, we can only have $F_{[3],rty}$ and $F_{[3],zty}$ non zero. Hence, the only relevant 2-form is F_{ty}.

Following this result and the above rules, the result of performing the first T duality in the direction y^1, which we denote by T(1), gives

T(1)

$$\begin{aligned} CF_{[4],y_1zty} &= F_{[3],zty} \\ CF_{[4],y_1rty} &= F_{[3],rty} \end{aligned} \tag{4.36}$$

T(2)

$$\begin{aligned} CF_{[5],y_2y_1zty} &= F_{[4],y_1zty} \\ CF_{[5],y_2y_1rty} &= F_{[4],y_1rty} \end{aligned} \tag{4.37}$$

T(3)

$$\begin{aligned} CF_{[6],y_3y_2y_1zty} &= F_{[5],y_2y_1zty} \\ CF_{[6],y_3y_2y_1rty} &= F_{[5],y_2y_1rty} \end{aligned} \tag{4.38}$$

T(4)

$$\begin{aligned} CF_{[7],y_4y_3y_2y_1zty} &= F_{[6],y_3y_2y_1zty} \\ CF_{[7],y_4y_3y_2y_1rty} &= F_{[6],y_3y_2y_1rty} \end{aligned} \tag{4.39}$$

We can now write

$$
\begin{aligned}
F_{[3]} &= \partial_a (C_{ty} dx^a \wedge dt \wedge dy) \\
&= \partial_r C_{ty} (dr \wedge dt \wedge dy) + \partial_z C_{ty} (dz \wedge dt \wedge dy)
\end{aligned}
\tag{4.40}
$$

Thus, we have

$$
\begin{aligned}
F_{[7],y_4 y_3 y_2 y_1 z t y} &= \partial_z C_{ty} \\
F_{[7],y_4 y_3 y_2 y_1 r t y} &= \partial_r C_{ty}
\end{aligned}
\tag{4.41}
$$

This new 3-form will correspond to the derivative of the final 2-form $C_{\mu\nu}$. Next, following along with Emparan and Elvang's results [5], we must take the Hodge dual of this 7-form. By doing this we obtain the (10-7)-form. This new 3-form will correspond to the derivative of the 2-form $C_{\mu\nu}$. This is needed for us to complete the final S-dual step which involves exchanging the $B_{\mu\nu}$ and $C_{\mu\nu}$ fields as previously described.

Hodge duals

By considering the Hodge dual of the $F_{[7]}$ forms, we can arrive at (3)-forms of the type we need. In general, we can write the components of the Hodge duals as follows:

$$
\begin{aligned}
&(\star F_{[7]})_{a_1 a_2 a_3} = \mathcal{E}^{b_4 b_5 b_6 b_7 b_8 b_9 b_{10}}_{a_1 a_2 a_3} \frac{F_{[7],b_4 b_5 b_6 b_7 b_8 b_9 b_{10}}}{7!} \\
&= \mathcal{E}^{b_4 b_5 b_6 b_7 b_8 b_9 b_{10}}_{a_1 a_2 a_3 c_4 c_5 c_6 c_7 c_8 c_9 c_{10}} G^{b_4 c_4} G^{b_5 c_5} G^{b_6 c_6} G^{b_7 c_7} G^{b_8 c_8} G^{b_9 c_9} G^{b_{10} c_{10}} \frac{F_{[7],b_4 b_5 b_6 b_7 b_8 b_9 b_{10}}}{7!} \\
&= \mathcal{E}_{a_1 a_2 a_3 c_4 c_5 c_6 c_7 c_8 c_9 c_{10}} G^{b_4 c_4} G^{b_5 c_5} G^{b_6 c_6} G^{b_7 c_7} G^{b_8 c_8} G^{b_9 c_9} G^{b_{10} c_{10}} F_{[7],b_4 b_5 b_6 b_7 b_8 b_9 b_{10}} \\
&= \mathcal{E}_{a_1 a_2 a_3 c_4 c_5 c_6 c_7 c_8 c_9 c_{10}} G^{y_4 c_4} G^{y_3 c_5} G^{y_2 c_6} G^{y_1 c_7} G^{r c_8} G^{t c_9} G^{y c_{10}} F_{[7],y_4 y_3 y_2 y_1 r t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 c_4 c_5 c_6 c_7 c_8 c_9 c_{10}} G^{y_4 c_4} G^{y_3 c_5} G^{y_2 c_6} G^{y_1 c_7} G^{z c_8} G^{t c_9} G^{y c_{10}} F_{[7],y_4 y_3 y_2 y_1 z t y} \\
&= \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 r c_9 c_{10}} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{rr} G^{t c_9} G^{y c_{10}} F_{[7],y_4 y_3 y_2 y_1 r t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 z c_9 c_{10}} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{zz} G^{t c_9} G^{y c_{10}} F_{[7],y_4 y_3 y_2 y_1 z t y} \\
&= \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 r t y} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{rr} G^{tt} G^{yy} F_{[7],y_4 y_3 y_2 y_1 r t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 z t y} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{zz} G^{tt} G^{yy} F_{[7],y_4 y_3 y_2 y_1 z t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 r y t} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{rr} G^{ty} G^{yt} F_{[7],y_4 y_3 y_2 y_1 r t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 z y t} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{zz} G^{ty} G^{yt} F_{[7],y_4 y_3 y_2 y_1 z t y} \\
&= \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 r t y} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{rr} (G^{tt} G^{yy} - (G^{ty})^2) F_{[7],y_4 y_3 y_2 y_1 r t y} \\
&+ \mathcal{E}_{a_1 a_2 a_3 y_4 y_3 y_2 y_1 z t y} G^{y_4 y_4} G^{y_3 y_3} G^{y_2 y_2} G^{y_1 y_1} G^{zz} (G^{tt} G^{yy} - (G^{ty})^2) F_{[7],y_4 y_3 y_2 y_1 z t y}
\end{aligned}
$$

Here $\mathcal{E}$ is the volume form tensor field, which has totally antisymmetric components and its only independent non vanishing component is $\mathcal{E}_{trz\phi_1\phi_2 yy_1y_2y_3y_4} = \sqrt{\det G} = \sqrt{G_{y_4y_4}G_{y_3y_3}G_{y_2y_2}G_{y_1y_1}G_{\phi_2\phi_2}G_{\phi_1\phi_1}G_{zz}G_{rr}(G_{tt}G_{yy} - G_{ty}^2)}$. Thus we can express the above term as:

$$
\begin{aligned}
(\star F_{[7]})_{z\phi_1\phi_2} &= \mathcal{E}_{z\phi_1\phi_2 y_4y_3y_2y_1rty}G^{y_4y_4}G^{y_3y_3}G^{y_2y_2}G^{y_1y_1}G^{rr}(G^{tt}G^{yy} - (G^{ty})^2)F_{[7],y_4y_3y_2y_1rty}\\
(\star F_{[7]})_{r\phi_1\phi_2} &= \mathcal{E}_{r\phi_1\phi_2 y_4y_3y_2y_1zty}G^{y_4y_4}G^{y_3y_3}G^{y_2y_2}G^{y_1y_1}G^{zz}(G^{tt}G^{yy} - (G^{ty})^2)F_{[7],y_4y_3y_2y_1zty}\\
(\star F_{[7]})_{z\phi_1\phi_2} &= \sqrt{\det G}G^{y_4y_4}G^{y_3y_3}G^{y_2y_2}G^{y_1y_1}G^{rr}(G^{tt}G^{yy} - (G^{ty})^2)F_{[7],y_4y_3y_2y_1rty}\\
(\star F_{[7]})_{r\phi_1\phi_2} &= \sqrt{\det G}G^{y_4y_4}G^{y_3y_3}G^{y_2y_2}G^{y_1y_1}G^{zz}(G^{tt}G^{yy} - (G^{ty})^2)F_{[7],y_4y_3y_2y_1zty}
\end{aligned}
\tag{4.42}
$$

Solving the system

Given the structure of the above definitions we can write the two main equations as

$$
\begin{aligned}
F_{[3],z\phi_1\phi_2} = (\star F_{[7]})_{z\phi_1\phi_2} &= \frac{-e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt} - G_{ty}^2}}\partial_r C_{ty}\\
F_{[3],r\phi_1\phi_2} = (\star F_{[7]})_{r\phi_1\phi_2} &= \frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt} - G_{ty}^2}}\partial_z C_{ty}
\end{aligned}
\tag{4.43}
$$

Here C_{ty} is given in equation (4.27). In particular, we need to outline some integrability conditions for our problem. We want to find the 2-form potential $C_{\mu\nu}$ corresponding to our new $F_{[3]}$, so that $F_{[3]} = dC_{[2]}$ defined above. From the form of (4.43) it makes sense to assume that the field $C_{[2]}$ is $C_{[2]} = f(r,z)d\phi_1 \wedge d\phi_2$ then by the commutativity of partial derivatives we know that $\partial_r\partial_z f = \partial_z\partial_r f$. We begin by using the value of C_{ty} given in (4.27) and substituting into (4.43) then simplifying using the values for $G_{11}, G_{yy}, G_{tt}, G_{ty}$ obtained previously. Thus, by comparing with (4.40) we can say that $F_{[3],r\phi_1\phi_2}$ corresponds with $\partial_r f$ and similarly $F_{[3],z\phi_1\phi_2}$ corresponds with $\partial_z f$ As a consequence, we need $\partial_r f = r\sinh(2\alpha)\partial_z U_0$ and $\partial_z f = -r\sinh(2\alpha)\partial_r U_0$. Using the knowledge that U_0 is a harmonic function, as well as the other properties of this problem, we must confirm this fact.

In order to show we can find a find a potential for $F_{[3]}$, we must check the necessary condition that F_3 is closed, $dF_3 = 0$. This requires that $\partial_z(F_{[3],r\phi_1\phi_2}) - \partial_r(F_{[3],z\phi_1\phi_2}) = 0$

$$\begin{aligned}
\partial_z\left(\frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-G_{ty}^2}}\partial_z C_{ty}\right) + \partial_r\left(\frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-C_{ty}^2}}\partial_r C_{ty}\right) = \\
\partial_r\left(\frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-G_{ty}^2}}\right)\partial_r C_{ty} + \frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-G_{ty}^2}}\partial_r^2 C_{ty} \\
+\partial_z\left(\frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-G_{ty}^2}}\right)\partial_z C_{ty} + \frac{e^{U_1}e^{U_2}}{G_{11}^2\sqrt{G_{yy}G_{tt}-G_{ty}^2}}\partial_z^2 C_{ty}
\end{aligned} \tag{4.44}$$

Through simplification via mathematica this reduces to

$$\begin{aligned}
e^{U_0+U_1+U_2}\sinh 2\alpha(U_{0,r}U_{1,r} + U_{0,r}U_{2,r} + U_{0,r}^2 + U_{0,rr}+ \\
U_{0,z}U_{1,z} + U_{0,z}U_{2,z} + U_{0,z}^2 + U_{0,zz})
\end{aligned} \tag{4.45}$$

Next, we will use the fact that $U_0 + U_1 + U_2 = \log r$. Thus $U_{0,z} + U_{1,z} + U_{2,z} = 0$ and $U_{0,r} + U_{1,r} + U_{2,r} = \frac{1}{r}$. Finally, we know that U_0 is a harmonic function so $U_{0,zz} + U_{0,rr} + \frac{U_{0,r}}{r} = 0$. Thus, we can rewrite the above equation as

$$\begin{aligned}
& e^{U_0+U_1+U_2}\sinh 2\alpha(U_{0,r}(U_{1,r} + U_{2,r} + U_{0,r}) + U_{0,rr}+ \\
& \qquad U_{0,z}(U_{1,z} + U_{2,z} + U_{0,z}) + U_{0,zz}) \\
& e^{U_0+U_1+U_2}\sinh 2\alpha(U_{0,r}\frac{1}{r} + U_{0,rr} + U_{0,z}(0) + U_{0,zz}) \\
& e^{U_0+U_1+U_2}\sinh 2\alpha(\frac{U_{0,r}}{r} + U_{0,rr} + U_{0,zz}) \\
& e^{U_0+U_1+U_2}\sinh 2\alpha(0) = 0
\end{aligned} \tag{4.46}$$

Thus, we can see that this represents an exact equation. So we can say with confidence that there exists a non-zero C field term, in particular, $C_{\psi\phi}$. We have shown that $F_{[3]}$ is closed, so that at least locally there must exist a $C_{[2]}$. However, finding it for a general U_0 is too difficult in general. We will determine C in explicit examples below.

4.3.3 Second S-Dual

In a process similar to the above outlined S-Dual section, we once again S-Dual on our new metric.

$$e^{\frac{\Phi}{2}} = \gamma^{\frac{1}{2}} \tag{4.47}$$

Thus, we obtain a similar result from before,

$$g_{ab} = \gamma^{\frac{-1}{2}} G_{ab} \tag{4.48}$$

Once again, we take the inverse of this dilaton to obtain the new scaling factor,

$$e^{\frac{\Phi'}{2}} = \gamma^{-\frac{1}{2}} \tag{4.49}$$

Then we find that this yields the same result as before, namely that

$$G'_{ab} = e^{\frac{\Phi'}{2}} g_{ab} = \gamma^{-1} G_{ab} \tag{4.50}$$

In particular, given our previously found version of the metric we can say that since,

$$G_{ab} dx^a dx^b = ds_{10}^2 = \gamma^{-1} ds_6^2 + \gamma \sum_{i=4} dy_i^2 \tag{4.51}$$

our new version of the metric must be

$$G'_{ab} dx^a dx^b = \gamma^{-1} G_{ab} = \gamma^{-2} ds_6^2 + \sum_{i=4} dy_i^2 \tag{4.52}$$

from (4.50) and where $ds'^2_{10} = G'_{ab} dx^a dx^b$. We may drop the prime notation on the new metric and simply write the new 10 dimensional line element as

$$ds_{10}^2 = \gamma^{-2} ds_6^2 + \sum_{i=4} dy_i^2 \tag{4.53}$$

Once again, we exchange $C_{\mu\nu}$ and $B_{\mu\nu}$ field terms

$$\begin{pmatrix} B'_{\mu\nu} \\ C'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} C_{\mu\nu} \\ -B_{\mu\nu} \end{pmatrix}. \tag{4.54}$$

In this case all the B field terms are identically 0, while the only non-zero C field term is the previously defined $C_{\phi_1\phi_2}$. Hence, we now have

$$C_{\mu\nu} = 0 \tag{4.55}$$

$$B_{\phi_1\phi_2} = C_{\phi_1\phi_2} = f \tag{4.56}$$

with all other terms being identically 0.

4.3.4 Second T-Dual and final Boost

Finally, we must perform one last T-dualization and a final boost. The T-dual process will follow through in the same way as defined before, thus we will simply highlight the relevant sections.

$$e^{\frac{-\Phi}{2}} = \gamma^{\frac{1}{2}} \tag{4.57}$$

$$G_{yy} \to \frac{1}{G_{yy}} \tag{4.58}$$

$$e^{2\Phi} \to \frac{e^{2\Phi}}{G_{yy}} = \frac{\gamma^{-2}}{G_{yy}} \tag{4.59}$$

$$G_{tt} \to G_{tt} - \frac{G_{ty}^2}{G_{yy}} = \frac{G_{tt}G_{yy} - G_{ty}^2}{G_{yy}} \tag{4.60}$$

$$B_{\phi_1\phi_2} \to B_{\phi_1\phi_2} \tag{4.61}$$

$$B_{ty} \to \frac{G_{ty}}{G_{yy}} \tag{4.62}$$

$$G_{ty} \to 0 \tag{4.63}$$

Then we finish by boosting the metric one last time

$$dt \to \cosh\delta\ dt + \sinh\delta\ dy \tag{4.64}$$

$$dy \to \sinh\delta\ dt + \cosh\delta\ dy. \tag{4.65}$$

Due to the complexity of these metric terms we will omit the specifics of the components when we define the new boosted metric.

$$\begin{aligned}
ds_6^2 \to\ & G_{tt}(\cosh^2\delta\ dt^2 + \sinh^2\delta\ dy^2 + \sinh 2\delta\ dtdy) + G_{\phi_1\phi_1}d\phi_1^2 \\
& + G_{\phi_2\phi_2}d\phi_2^2 + G_{rr}(dr^2 + dz^2) + G_{yy}(\cosh^2\delta\ dy^2 + \sinh^2\delta\ dt^2 + \sinh 2\delta\ dtdy) \\
& = (G_{yy}\sinh^2\delta\ + G_{tt}\cosh^2\delta)dt^2 + G_{\phi_1\phi_1}d\phi_1^2 + G_{\phi_2\phi_2}d\phi_2^2 + G_{rr}(dr^2 + dz^2) \\
& + (G_{yy}\cosh^2\delta\ + G_{tt}\sinh^2\delta)dy^2 + (G_{yy} + G_{tt})\sinh 2\delta\ dtdy.
\end{aligned} \tag{4.66}$$

Writing this to isolate for the y and t terms, we can say

$$\begin{aligned}
ds_6^2 = & (G_{yy}\cosh^2\delta\ + G_{tt}\sinh^2\delta)\left(dy + \frac{(G_{yy} + G_{tt})\sinh 2\delta}{2(G_{yy}\cosh^2\delta\ + G_{tt}\sinh^2\delta)}dt\right)^2 \\
& + \frac{G_{tt}G_{yy}}{G_{tt}\sinh^2\delta + G_{yy}\cosh^2\delta}dt^2 + G_{\phi_1\phi_1}d\phi_1^2 + G_{\phi_2\phi_2}d\phi_2^2 + G_{rr}(dr^2 + dz^2)
\end{aligned} \tag{4.67}$$

where this is a solution to the 6 dimensional action

$$S_6 = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-\hat{g}}e^{-2\Phi}\left(R^{(6)} + (4\nabla\Phi)^2 - \frac{1}{12}\hat{H}^2\right) \tag{4.68}$$

A Kaluza-Klein reduction in the y-direction yield the result

$$\hat{g}_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}\left(dy + A_\mu^{(n)}dx^\mu\right)^2 \tag{4.69}$$

where $x^M = (x^\mu, y)$ and $e^{2\sigma} = g_{yy}$ leads to the 5 dimensional action

$$\begin{aligned}
S_5 = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}e^{-2\Phi+\sigma}(& R^{(5)} + (4\nabla\Phi)^2 - 4\nabla\Phi\nabla\sigma - \frac{1}{12}H^2 \\
& -\frac{1}{4}e^{2\sigma}\left(F^{(n)}\right)^2 - \frac{1}{4}e^{-2\sigma}\left(F^{(1)}\right)^2).
\end{aligned} \tag{4.70}$$

Here we are using the notation that $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$, $A_\mu^{(1)} = \hat{B}_{\mu y}$, $\hat{H_{y\mu\nu}} = F_{\mu\nu}^{(1)}$, and $H_{\mu\nu\rho} = \left(\partial_{[\mu}\hat{B}_{\nu\rho]} - A_{[\mu}^{(n)}F_{\nu\rho]}^{(1)}\right)$. As well, the terms κ_6 and κ_5 are coupling constants and do not play a major role in the calculations. See Emparan and Elvang's work for full details of this action [5].

Referencing Emparan and Elvang [5] we can say the following terms can be read off from the above system

$$\begin{aligned}
A_t^{(n)} &= \frac{(G_{yy} + G_{tt})\sinh 2\delta}{2(G_{yy}\cosh^2\delta \ + G_{tt}\sinh^2\delta)} \\
&= \frac{\left(-1 + e^{2U_0}\right)\cosh\delta\sinh\delta}{-\cosh^2\delta + e^{2U_0}\sinh^2\delta} \\
e^{2\sigma} &= G_{yy}\cosh^2\delta \ + G_{tt}\sinh^2\delta \\
&= \frac{\cosh^2\delta - e^{2U_0}\sinh^2\delta}{\cosh^2\beta - e^{2U_0}\sinh^2\beta} \\
g_{tt} &= \frac{G_{tt}G_{yy}}{G_{tt}\mathrm{sinh}^2\delta + G_{yy}\mathrm{cosh}^2\delta} \\
&= \frac{-e^{2U_0}}{\left(-\cosh^2\beta + e^{2U_0}\sinh^2\beta\right)\left(-\cosh^2\delta + e^{2U_0}\sinh^2\delta\right)} \\
g_{\phi_1\phi_1} &= G_{\phi_1\phi_1} = \gamma^{-2}e^{U_1} \\
g_{\phi_2\phi_2} &= G_{\phi_2\phi_2} = \gamma^{-2}e^{U_2} \\
g_{rr} &= G_{rr} = \gamma^{-2}e^{2\nu} \\
g_{zz} &= G_{rr} = \gamma^{-2}e^{2\nu}
\end{aligned} \tag{4.71}$$

From Emparan and Elvang [5], we can also read off the additional field term A_t^1 as

$$A_t^1 = B_{ty} = \frac{\left(-1 + e^{2U_0}\right)\cosh\beta\sinh\beta}{-\cosh^2\beta + e^{2U_0}\sinh^2\beta} \tag{4.72}$$

$$\tag{4.73}$$

As well, we can confirm that $A_t^{(n)} = A_t^1$ when we set $\beta = \delta$ which is expected from Emparan and Elvang [5]. In addition, we can confirm that $\sigma \to 0$ when $\alpha = \beta = \delta$

$$\begin{aligned}
e^{2\sigma} &= \frac{\cosh^2\delta - e^{2U_0}\sinh^2\delta}{\cosh^2\beta - e^{2U_0}\sinh^2\beta} \\
&\to \frac{\cosh^2\beta - e^{2U_0}\sinh^2\beta}{\cosh^2\beta - e^{2U_0}\sinh^2\beta} = 1
\end{aligned} \tag{4.74}$$

As well, we can define the final field term as outlined in Emparan and Elvang [5] to be

$$\begin{aligned} A_t^5 &= A_t^1 \ [\alpha \leftrightarrow \beta] \\ A_t^5 &= \frac{\left(-1 + e^{2U_0}\right) \cosh\alpha \sinh\alpha}{-\cosh^2\alpha + e^{2U_0}\sinh^2\alpha} \end{aligned} \tag{4.75}$$

The effective dilaton can be written as $\Phi_{\text{eff}} = \Phi - \frac{\sigma}{2}$. Thanks to writing this dilaton in this form, we can write the Einstein metric in terms of the string metric as $g^E_{\mu\nu} = e^{\frac{-4}{3}\Phi_{\text{eff}}} G_{\mu\nu} = e^{\frac{-4}{3}\Phi} e^{\frac{2}{3}\sigma} G_{\mu\nu}$. More concretely we can say

$$\begin{aligned} g^E_{\mu\nu} &= e^{\frac{-4}{3}\Phi} e^{\frac{2}{3}\sigma} G_{\mu\nu} \\ &= (G^2_{yy}\gamma^4 \frac{\cosh^2\delta - e^{2U_0}\sinh^2\delta}{\cosh^2\beta - e^{2U_0}\sinh^2\beta})^{\frac{1}{3}} G_{\mu\nu} \\ &= \left(\frac{(\cosh^2\beta - e^{2U_0}\sinh^2\beta)(\cosh^2\delta - e^{2U_0}\sinh^2\delta)}{(\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)} \right)^{\frac{1}{3}} G_{\mu\nu} \end{aligned} \tag{4.76}$$

For simplicity, we can denote $\Delta = (\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)(\cosh^2\beta - e^{2U_0}\sinh^2\beta)(\cosh^2\delta - e^{2U_0}\sinh^2\delta)$. In particular, we are interested in the case of $\alpha = \beta = \delta$. This case yields a solution to a minimal 5 dimensional supergravity [5]. Thus, we can write $\Delta = (\cosh^2\alpha - e^{2U_0}\sinh^2\alpha)^3$. Then we can now write the Einstein metric as

$$g^E = ds_5^2 = -e^{2U_0}\Delta^{\frac{-2}{3}} dt^2 + \Delta^{\frac{1}{3}}(e^{2U_1}d\phi_1^2 + e^{2U_2}d\phi_2^2 + e^{2\nu}(dz^2 + dr^2)) \tag{4.77}$$

In addition, since $\alpha = \beta = \delta$ we know that $A_t^1 = A_t^5 = A_t^{(n)}$ which we can refer to simply as A. We want to define the fields A so that they are similar to the form of the action [5]. To do this we simply scale the field(s) as

$$\bar{A} = \frac{\sqrt{3}}{2} A \tag{4.78}$$

This gives us the appropriate scaling factor needed to solve the field equations (2.10) of Einstein-Maxwell theory [15]

$$S_{EM} = \int_M \sqrt{-g}\left(R - F_{ab}F^{ab}\right) d^5x \tag{4.79}$$

where $F_{ab} = \partial_a \bar{A}_b - \partial_b \bar{A}_a$. The field equations come from extremizing the action above. Knowing this, we can also say what the electric charge is for this system. It is known that q is the coefficient for the second order expansion of $\bar{A}$ and that $Q = \pi q$. The expansion for $\bar{A}$ is

$$\bar{A} = \frac{\sqrt{3}}{2}\frac{a\cosh\alpha\sinh\alpha}{R^2} + \mathcal{O}(R^{-4}). \tag{4.80}$$

This comes from the series representation found in 4.75. This result works for both cases we outlined previously, that is the symmetric and asymmetric cases in Chapter 3.

The definition of the electric charge is $Q = \frac{1}{4\pi}\int_{S^3_\infty} *F$. The easiest way to calculate this is to note that $Q = \pi q$ where q is the coefficient of the $\frac{1}{R^2}$ term in the asymptotic expansion of $\bar{A}$. In our case, $q = \frac{\sqrt{3}a\cosh\alpha\sinh\alpha}{2}$ and therefore

$$Q = \frac{\pi\sqrt{3}a\cosh\alpha\sinh\alpha}{2}. \tag{4.81}$$

This defines the electric charge for our boosted and transformed system. This applies to both the symmetric and asymmetric case as the $\bar{A}$ only depends on U_0 which is the same in both cases.

Next, we want to study the conical singularities of this new metric. In general, we will write that the form of the metric goes as

$$g = -e^{2\bar{U}_0}dt^2 + e^{2\bar{U}_1}d\phi_1^2 + e^{2\bar{U}_2}d\phi_2^2 + e^{2\bar{\nu}}(dz^2 + dr^2) \tag{4.82}$$

In our particular case, we have that

$$e^{2\bar{U}_0} = \Delta^{\frac{-2}{3}}e^{2U_0} \tag{4.83}$$

$$e^{2\bar{U}_1} = \Delta^{\frac{1}{3}}e^{2U_1} \tag{4.84}$$

$$e^{2\bar{U}_2} = \Delta^{\frac{1}{3}}e^{2U_2} \tag{4.85}$$

$$e^{2\bar{\nu}} = \Delta^{\frac{1}{3}}e^{2\nu} \tag{4.86}$$

Given the previously defined form for the conical singularities, we can clearly see that in the regions where U_1 and U_2 are defined, the ratios will be the same and the conditions on the singularities are unchanged. Both cases are outlined below. First we examine the case where the symmetry axis $\frac{\partial}{\partial \phi_1}$ vanishes,

$$\Delta\eta_1 = 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 e^{2\bar{\nu}}}{e^{2\bar{U}_1}}} \tag{4.87}$$

$$\Delta\eta_1 = 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 \Delta^{\frac{1}{3}} e^{2\nu}}{\Delta^{\frac{1}{3}} e^{2U_1}}} \tag{4.88}$$

$$= 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 e^{2\nu}}{e^{2U_1}}}. \tag{4.89}$$

Similarly, we can find the result of the U_2 case

$$\Delta\eta_2 = 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 e^{2\bar{\nu}}}{e^{2\bar{U}_2}}} \tag{4.90}$$

$$\Delta\eta_2 = 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 \Delta^{\frac{1}{3}} e^{2\nu}}{\Delta^{\frac{1}{3}} e^{2U_2}}} \tag{4.91}$$

$$= 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 e^{2\nu}}{e^{2U_2}}}. \tag{4.92}$$

However, in the region where U_0 is defined, this is the Horizon region, the ratios change. In this case the metric at the horizon is $ds^2 = \Delta^{1/3}(e^{2U_1} d^2\phi_1 + e^{2U_2} d^2\phi_2 + e^{2\nu} dz^2)$ as t and r are constants. Specifically, we can say that

$$\Delta\eta_0 = 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 \Delta^{\frac{1}{3}} e^{2\nu}}{\Delta^{\frac{-2}{3}} e^{2U_0}}} \tag{4.93}$$

$$= 2\pi \lim_{r\to\infty} \sqrt{\frac{r^2 \Delta e^{2\nu}}{e^{2U_0}}} \tag{4.94}$$

And to a first order approximation, this is simply

$$\Delta\eta_0 = 2\sqrt{2}\pi \cosh^3\alpha \sqrt{\frac{ab(a-c)}{(b-c)^2}} \tag{4.95}$$

So we can see that the boosts will scale $\Delta\eta_0$ by a factor of $\cosh^3\alpha$ versus the non-boosted case discussed above.

Moreover, we can say that the temperature and surface gravity of the horizon are related to the conical deficit [7, 11]. Specifically they are defined by

$$\kappa = 2\pi T = \frac{2\pi}{\Delta\eta_0} = \frac{1}{\sqrt{2}\cosh^3\alpha}\sqrt{\frac{(b-c)^2}{ab(a-c)}} \tag{4.96}$$

$$Area_H = \frac{4\pi^2 L_{rod}}{\kappa} \tag{4.97}$$

For the symmetric case, we know that conical singularities, $\Delta\eta$ are $\Delta\eta_1 = \sqrt{\frac{b(b-a)}{(b-c)^2}}$ and $\Delta\eta_2 = \sqrt{\frac{c(c-a)}{(b-c)^2}}$. In this, the area of the U_1 and U_2 bubbles are simply

$$Area_1 = \int\int \sqrt{g}dzd\phi_2$$
$$Area_2 = \int\int \sqrt{g}dzd\phi_1$$

Using this information we can simplify the integral term $\int d\phi_i$ to be $\Delta\eta_i$. Thus giving area integral values of

$$Area_1 = \Delta\eta_2 \int \sqrt{g}dz = \Delta\eta_2 \int \Delta^{\frac{1}{3}} e^{U_2+\nu}dz$$
$$Area_2 = \Delta\eta_1 \int \sqrt{g}dz = \Delta\eta_1 \int \Delta^{\frac{1}{3}} e^{U_1+\nu}dz$$

and through the methods outlined in Chapter 3 (3.4-3.5) then we know ν can be given by (3.42).

In the asymmetric case the $\Delta\eta$ terms are $\Delta\eta_1 = \sqrt{\frac{b(b-a)}{c(c-a)}}$ and $\Delta\eta_2 = \sqrt{\frac{(c-b)^2}{c(c-a)}}$ and we know ν can be given by (3.46).

$$Area_1 = \Delta\eta_2 \int \sqrt{g}dz = \Delta\eta_2 \int \Delta^{\frac{1}{3}} e^{U_2+\nu}dz$$
$$Area_2 = \Delta\eta_1 \int \sqrt{g}dz = \Delta\eta_1 \int \Delta^{\frac{1}{3}} e^{U_1+\nu}dz$$

These expressions give the area for each of these bubbles in general terms via their integral. Unfortunately neither of these integrals can be evaluated to find exact solutions due to the complexity of the e^{ν} factors.

In addition, we can discuss the mass of the horizon. In general the series expansion of the g_{tt} term will look like

$$g_{tt} = -1 + \frac{8M}{3\pi R^2} + \mathcal{O}(R^{-4}) \tag{4.98}$$

where M is our mass. In the case outlined above the g_{tt} will look like

$$g_{tt} = -1 + \frac{a\cosh 2\alpha}{R^2} + \mathcal{O}(R^{-4}) \tag{4.99}$$

Thus, in our case we have that $M = \frac{3}{8}a\pi\cosh 2\alpha$. It is valuable to compare this result to the Smarr relation [15]. This is general form can be expressed as

$$M' = \frac{3\kappa A_H}{16\pi} + \Phi_H Q \tag{4.100}$$

Here, Φ_H is the electric potential on the horizon. In the simple pure vacuum case where $Q = 0$, the Smarr relation suggests that $M' = \frac{3\kappa A_H}{16\pi} = \frac{12\pi^2 L_{rod}}{16\pi}$ using the relationship in (4.97). In each case we are considering the $L_{rod} = a$ yields the result $M' = \frac{3\pi a}{4}$. In our result we can set $\alpha = 0$ and see that we have obtained $M = \frac{3a\pi}{8}$. Thus, these results differ by a factor of 2. This is believed to be a result of the associated bubbles in our construction. In summary, we have found a solution to the Einstein-Maxwell equations with the forms g^E and $\bar{A}$.

Chapter 5

Conclusions

In 4 dimensional general relativity we are able to characterize a black hole via three of its properties. Specifically, those properties are the black hole's mass, M, charge, Q, and angular momentum, J. This is known as the "no hair" theorem [3]. The key take-away from this work is that two black holes with the same properties are identical and belong to the Kerr-Newman family of solutions [13, 20].

Unfortunately, as we extend to higher dimensions we lose this sense of uniqueness. It may be possible that so -called "bubbles" or 2-cycles may exist. These are closed 2 dimensional surfaces which can carry positive energy without the need for a black hole [10, 17]. Thus, we may observe two black holes with the same properties (M, Q, J) but we are unable to determine if they are the same black hole or if the existence of these bubbles have influenced these observed properties.

In this work several tools and methods were employed in order to undertake an analysis of the non-uniqueness of these higher dimensional systems. In particular the use of Weyl solutions was of great importance. These solutions provided general static axisymmetric solutions of the vacuum Einstein equations [6, 28]. This formulation is extremely useful for this work as it allows for the explicit manipulation of the solution to the associated Laplace's equation which correspond to our system. This is done in concert with the so-called rod diagrams for visualization purposes.

In this work solutions were constructed that included the existence of bubbles in two different killing vector directions. These solutions are asymptotically flat and correspond to both symmetric and asymmetric examples Figs 3.3, 3.4. Using the

Weyl solution techniques it was possible to solve for the vacuum solutions explicitly and check for their regularity.

Through this work, charged solutions of the Einstein-Maxwell theory were constructed with the aim of resolving all singularities. However, through further analysis it was clear that these singularities remain. While this work was being completed it was proven that there are no static Einstein-Maxwell black holes with solitons [18]. This was verified explicitly in our particular cases. As well, the expected mass from the Smarr relation was checked and determined to differ by a factor of 2 from the anticipated value of $M = \frac{3a\pi}{8}$. It is believed that this is due to the existence of bubbles and singularities in this system. This was not obvious prior to explicit calculation and, in fact, the Smarr relation may hold even when there are singularities.

In the future, research may be conducted on including a rotating component into the analysis. Recent work by Figueras et al [8] may serve as a good starting point. This work incorporates inverse scattering theory in charged rotating black holes to solve Einstein-Maxwell equations. At the time of the completion of this work no known solutions have been found via these methods.

Bibliography

[1] G. Alekseev and V. Belinsky. Static gravitational solitons. *Zh. Eksp. Teor. Fiz*, 78:1297–13, 1980.

[2] B. Carter. Axisymmetric black hole has only two degrees of freedom. *Physical Review Letters*, 26(6):331, 1971.

[3] P. T. Chrusciel, J. Lopes Costa, and M. Heusler. Stationary Black Holes: Uniqueness and Beyond. *Living Rev. Rel.*, 15:7, 2012.

[4] A. Einstein. The foundation of the general theory of relativity. *Annalen Phys.*, 14:769–822, 1916.

[5] H. Elvang and R. Emparan. Black rings, supertubes, and a stringy resolution of black hole non-uniqueness. *Journal of High Energy Physics*, 2003(11):035, 2003.

[6] R. Emparan and H. S. Reall. Generalized weyl solutions. *Physical Review D*, 65(8):084025, 2002.

[7] R. Emparan and H. S. Reall. Black holes in higher dimensions. *Living Reviews in Relativity*, 11(1):6, 2008.

[8] P. Figueras, E. Jamsin, J. V. Rocha, and A. Virmani. Integrability of Five Dimensional Minimal Supergravity and Charged Rotating Black Holes. *Class. Quant. Grav.*, 27:135011, 2010.

[9] G. W. Gibbons, D. Ida, and T. Shiromizu. Uniqueness and nonuniqueness of static vacuum black holes in higher dimensions. *Prog. Theor. Phys. Suppl.*, 148:284–290, 2003.

[10] S. Gunasekaran, U. Hussain, and H. K. Kunduri. Soliton mechanics. *Physical Review D*, 94(12):124029, 2016.

[11] T. Harmark. Stationary and axisymmetric solutions of higher-dimensional general relativity. *Physical Review D*, 70(12):124002, 2004.

[12] H. Iguchi and T. Mishima. Solitonic generation of vacuum solutions in five-dimensional general relativity. *Physical Review D*, 74(2):024029, 2006.

[13] W. Israel. Event horizons in static vacuum space-times. *Physical review*, 164(5):1776, 1967.

[14] M. Kenmoku, S. Uchida, and T. Matsuyama. Conical singular solutions in (2+1)-dimensional gravity employing the adm canonical formalism. 2003.

[15] H. K. Kunduri and J. Lucietti. Electrically charged dilatonic black rings. *Physics Letters B*, 609(1-2):143–149, 2005.

[16] H. K. Kunduri and J. Lucietti. Black hole non-uniqueness via spacetime topology in five dimensions. *Journal of High Energy Physics*, 2014(10):82, 2014.

[17] H. K. Kunduri and J. Lucietti. The first law of soliton and black hole mechanics in five dimensions. *Classical and Quantum Gravity*, 31(3):032001, 2014.

[18] H. K. Kunduri and J. Lucietti. No static bubbling spacetimes in higher dimensional einstein–maxwell theory. *Classical and Quantum Gravity*, 35(5):054003, 2018.

[19] D. Marolf. Black holes and branes in supergravity. *arXiv preprint arXiv:1107.1022*, 2011.

[20] S. D. Mathur. Gian course on the black hole information paradox, June-July 2016.

[21] T. Müller and F. Grave. Catalogue of spacetimes. *arXiv preprint arXiv:0904.4184*, 2009.

[22] R. C. Myers. Dielectric-branes. *Journal of High Energy Physics*, 1999(12):022, 2000.

[23] D. C. Robinson. Uniqueness of the kerr black hole. *Physical Review Letters*, 34(14):905, 1975.

[24] R. Schon and S.-T. Yau. On the Proof of the positive mass conjecture in general relativity. *Commun. Math. Phys.*, 65:45–76, 1979.

[25] M. S. Volkov and D. V. Gal'tsov. Gravitating nonAbelian solitons and black holes with Yang-Mills fields. *Phys. Rept.*, 319:1–83, 1999.

[26] R. M. Wald. General relativity(book). *Chicago, University of Chicago Press, 1984, 504 p*, 1984.

[27] S. Weinberg. *Gravitation and cosmology: principles and applications of the general theory of relativity*, volume 1. Wiley New York, 1972.

[28] H. Weyl. Zur gravitationstheorie. *Annalen der Physik*, 359(18):117–145, 1917.